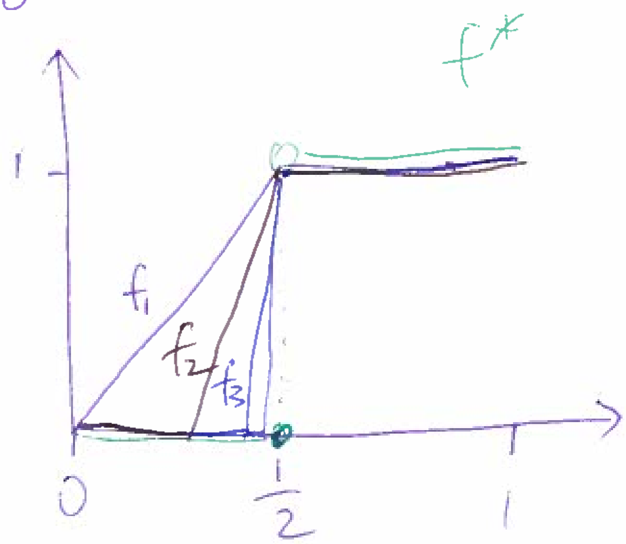
Back to Completeness

$(CB([0,1]), d_1)$ is <u>not</u> complete (again!)

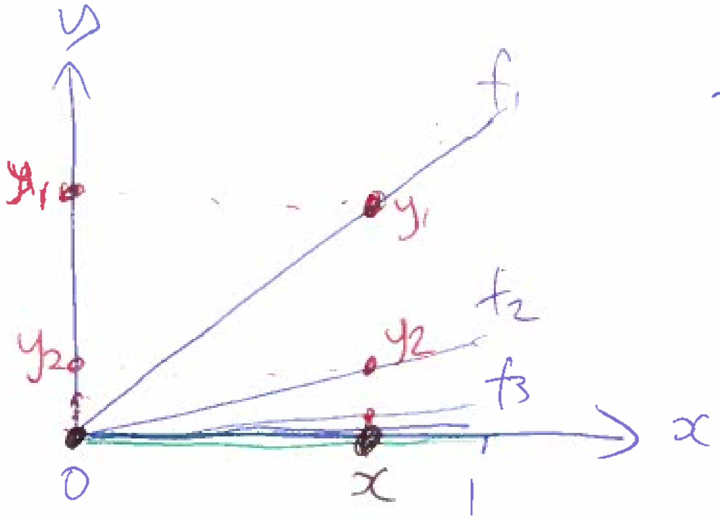


Notice that $d_1(f_n, f_m)$ approaches 0 as n, m get bigger (i.e. $n, m > N$). So f_n is a Cauchy sequence.

Note that $f^* \notin CB([0,1])$.
In fact there is no function inside $CB([0,1])$ such that f_n ~~appr~~ converges to it. So $(CB([0,1]), d_1)$ is <u>not</u> complete.

<u>Theorem C.14</u> Let (X, d_X) and (Y, d_Y) be metric spaces. If (Y, d_Y) is a complete metric space, then $(B(X,Y), d_\infty)$ and $(CB(X,Y), d_\infty)$ are complete metric spaces.

<u>Proof</u> Let f_n be a Cauchy sequence in $(B(X,Y), d_\infty)$. Then for any $x \in X$, the sequence $y_n = f_n(x)$ is a Cauchy sequence inside (Y, d_Y).



$f_n(x) = \frac{x}{n}$.

$y_n = f_n(x)$

f^*

Since (Y, d_Y) is complete, y_n is convergent. Let's call the limit $f^*(x)$. So we have defined a function $f^* : X \to Y$.

Next step: prove $d_\infty(f_n, f^*) \to 0$.

By continuity of d_Y,

$$d_Y(f^*(x), f_n(x)) = \lim_{m \to \infty} d_Y(f_m(x), f_n(x)),$$

for all $x \in X$. Taking the supremum of RHS ~~both sides~~, we get

$$d_Y(f^*(x), f_n(x)) \le \lim_{m \to \infty} d_\infty(f_m, f_n).$$

So,

$$d_\infty(f^*, f_n) \le \lim_{m \to \infty} d_\infty(f_m, f_n).$$

Since f_n is a Cauchy sequence, the right side converges to zero as $n \to \infty$.

So the left side converges to zero, i.e. $d_\infty(f^*, f_n) \to 0$.

Next step: $f^* \in B(X, Y)$ $(\mathbb{R}_+, d_2)$

Since $d_\infty(f^*, f_n) \to 0$, there exists some M s.t. $d_\infty(f^*, f_M) < 1$. Since $f_M \in B(X, Y)$, there exists an open ball $B_r(y)$ s.t. $f_M(X) \subseteq B_r(y)$. By the triangle inequality,

$$d_Y(f^*(x), y) \leq d_Y(f^*(x), f_M(x)) + d_Y(f_M(x), y)$$

$$< 1 + r,$$

We conclude that $f^*(X) \subseteq B_{1+r}(y)$.
So f^* is bounded, i.e. $f^* \in B(X, Y)$.
So $f_n \to f^*$, and hence $(B(X, Y), d_\infty)$ is a complete metric space.

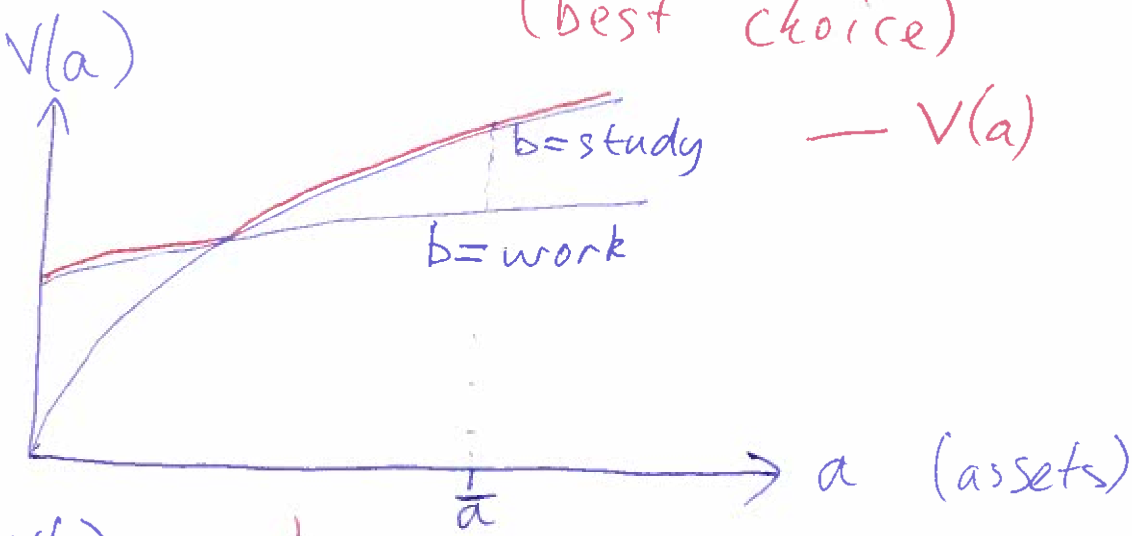
Skip second half... $\square$

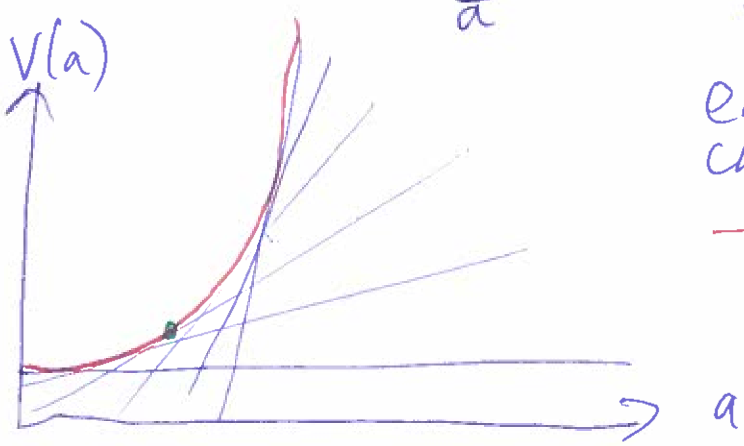
2.3 Upper Envelopes

How can we differentiate

$$\underbrace{V(\underbrace{a}_{\text{state variable}})}_{\text{value function}} = \max_{\underbrace{b}_{\text{choice variable}}} \underbrace{v(a,b)}_{\text{objective function}}$$

$$= v(a, \underbrace{b(a)}_{\text{policy function (best choice)}})$$





each — is a choice

— V(a)

Theorem 2.1 (Envelope Theorem)

Let $v : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a differentiable function, $V(a) = \max_{b \in \mathbb{R}^m} v(a,b)$, and let $b(a)$ be its policy function. If V is differentiable function, then

$$V'(a) = \frac{\partial v(a,b)}{\partial a}\Big|_{b=b(a)}.$$

I didn't write

$$\frac{\partial v(a, b(a))}{\partial a}$$

Proof Fix a particular state $\bar a$.

The value function of a "lazy" decision maker who choose $b(\bar a)$ regardless of what the state a actually is.

Their value function is

$$L(a) = v(a, b(\bar a)).$$ wrong choice

Notice that $L(a) \le V(a)$ for all a, and $L(\bar a) = V(\bar a)$. So $\bar a$ solves

$$\min_a V(a) - L(a).$$ FOC: $V'(\bar a) - L'(\bar a) = 0$

FOC: $V'(\bar a) = L'(\bar a)$.

Now $L'(\bar a) = \cancel{\partial v(a, b(\bar a))}\ \frac{\partial v(a,b)}{\partial a}\Big|_{\substack{a=\bar a \\ b=b(\bar a)}}.$

So $V'(\bar{a}) = \left.\dfrac{\partial v(a,b)}{\partial a}\right|_{\substack{a=\bar{a} \\ b=b(\bar{a})}}$.

Rewrite: $V'(a) = \left.\dfrac{\partial v(a,b)}{\partial a}\right|_{b=b(a)}$. $\square$

e.g. ℓ # workers
w wages

production function

$$\pi(w) = \max_{\ell} 10\sqrt{\ell} - w\ell.$$

Q: What is $\pi'(w)$?

A: with envelope theorem

$$\pi'(w) = \left[\frac{\partial}{\partial w}\left\{10\sqrt{\ell} - w\ell\right\}\right]_{\ell=\ell(w)}$$

$$= \left[-\ell\right]_{\ell=\ell(w)}$$

$$= -\ell(w).$$

A: without envelope theorem

① calculate $\ell(w)$. FOC ℓ:

$$10\frac{1}{\sqrt{\ell}}\frac{1}{2} - w = 0$$

$$\Leftrightarrow \quad \frac{5}{\sqrt{\ell}} = w$$

$$\Leftrightarrow \quad \sqrt{\ell} = \frac{5}{w}$$

$$\Leftrightarrow \quad \ell(w) = \frac{25}{w^2}.$$

② Subst: $\pi(w) = 10\sqrt{\ell(w)} - w\,\ell(w)$

$$= 10\sqrt{\frac{25}{w^2}} - w\frac{25}{w^2}$$

$$= 10\cdot\frac{5}{w} - \frac{25}{w}$$

$$= \frac{25}{w}.$$

(3) Diff: $\pi'(w) = -\frac{25}{w^2}$

$= -\ell(w).$

Recall: $\pi(p;w) = \max_{x \in \mathbb{R}^n_+} pf(x) - w \cdot x.$

Apply envelope theorem:

$$\frac{\partial \pi(p;w)}{\partial p} = \left[\frac{\partial}{\partial p}\left\{pf(x) - w \cdot x\right\}\right]_{x=x(p;w)}$$

$$= \left[f(x)\right]_{x=x(p;w)}$$

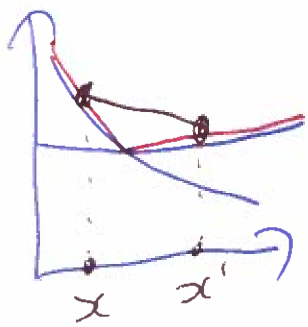
$$= f(x(p;w)). \leftarrow \text{output}$$

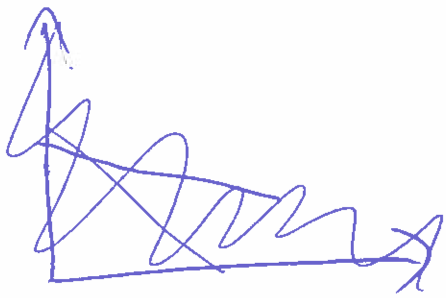
$$\frac{\partial \pi(p;w)}{\partial w_i} = \left[\frac{\partial}{\partial w_i}\left\{pf(x) - w \cdot x\right\}\right]_{x=x(p;w)}$$

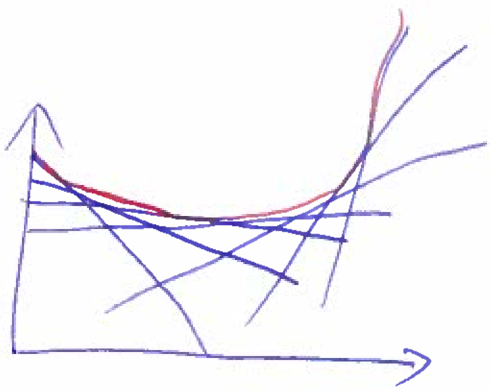
$$= \left[-x_i\right]_{x=x(p;w)}$$

$$= -x_i(p;w).$$

<u>Theorem 2.2</u> Suppose $v(a,b)$ is convex in a. Then $V(a) = \max_b v(a,b)$ is convex.







Proof Recall V is convex if

$$\underbrace{tV(a)+(1-t)V(a')}_{\text{line}} \geq \underbrace{V(ta+(1-t)a')}_{\text{curve}}$$

for all a, a' and all $t \in [0,1]$.

Now,

$$\begin{aligned}
& tV(a)+(1-t)V(a') \\
&= t\,v(a, b(a)) + (1-t)\,v(a', b(a')) \\
&\geq t\,v(a, \underbrace{b(ta+(1-t)a')}_{\text{"wrong" choice}}) + (1-t)\,v(a', b(a')) \\
&\geq t\,v(a, b(ta+(1-t)a')) \\
&\quad + (1-t)\,v(a', \underbrace{b(ta+(1-t)a')}_{\text{same wrong choice}}) \\
&\geq v(ta+(1-t)a', b(ta+(1-t)a')) \\
&= V(ta+(1-t)a').
\end{aligned}$$

since v is convex in a.

$\square$