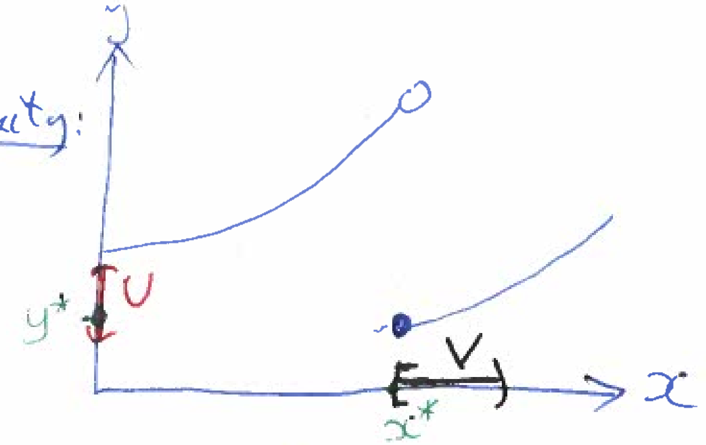
Proof ~~f is~~ seq. continuous $\Rightarrow$ open set continuity:



Let U be any open set in (Y, d_Y), and let $V = f^{-1}(U)$. We need to prove that V is an open set inside (X, d_X).

Pick any $x \in V$. We just need to prove that x is an interior point of V. Let $y = f(x)$. Since U is open and $y \in U$, there is some open ball $B_s(y) \subseteq U$. By Theorem C.7 (open ball continuity), there is some ball $B_r(x)$ such that $f(B_r(x)) \subseteq B_s(y)$. So

$$f(B_r(x)) \subseteq B_s(y) \subseteq U.$$

$$\Rightarrow f^{-1}(f(B_r(x))) \subseteq f^{-1}(B_s(y)) \subseteq f^{-1}(U).$$

Also, $B_r(x) \subseteq f^{-1}(f(B_r(x)))$.

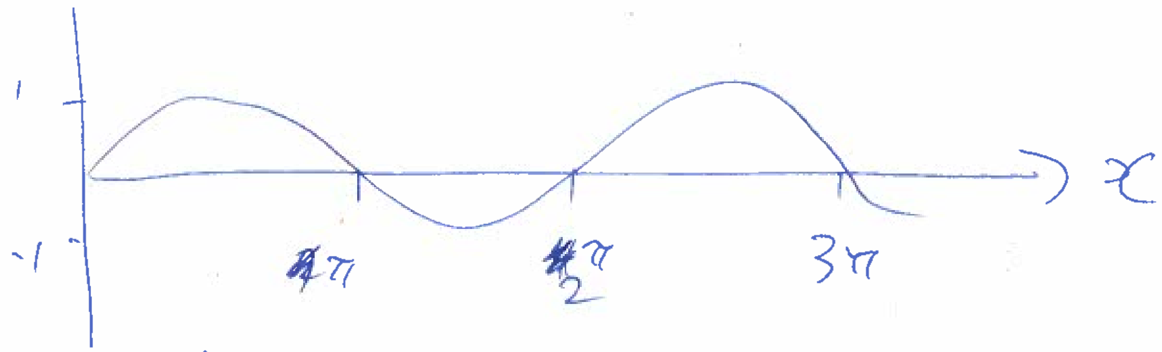
So $B_r(x) \subseteq f^{-1}(f(B_r(x))) \subseteq f^{-1}(B_s(y)) \subseteq f^{-1}(U) = V$

and hence $B_r(x) \subseteq V$. So x is an interior point of V.

open set continuity $\Rightarrow$ seq. continuity:

Pick any $x \in X$, and let $y = f(x)$, and pick any open ball $U = B_s(y)$. Since U is an open set in (Y, d_Y), we assumed $f^{-1}(U)$ is an open set in (X, d_X). Since $f^{-1}(U)$ is open and $x \in f^{-1}(U)$, there exists some ball $B_r(x) \subseteq f^{-1}(U)$. So $f(B_r(x)) \subseteq U = B_s(y)$.

So f is continuous by Theorem C.7 $\square$



$f(x) = \sin x$

$A = [0, \pi]$, $\quad f(A) = [0,1]$, $\quad f^{-1}(f(A)) = \bigcup_{n\in\mathbb{Z}}^{\infty} [2n\pi, (2n+1$

$B = [0,1]$ $\quad f^{-1}(B) = f^{-1}(f(A))$

$$f(f^{-1}(B)) = [0,1].$$

C.7 Completeness

Consider $A = [0,1)$. A is not closed in $(\mathbb{R}, d_2)$.
But A is closed in (A, d_2).

<u>Def</u> Let (X, d) be a metric space. A sequence $x_n \in X$ is called a <u>Cauchy sequence</u> if for every radius $r > 0$, there exists some N such that for all $n, m > N$

$$d(x_n, x_m) < r.$$

<u>Def</u> A metric space (X, d) is <u>complete</u> if every Cauchy sequence x_n is convergent, i.e. converges to some $x^* \in X$.

eg: * $([0,1), d_2)$ is <u>not</u> a complete metric space, because $x_n = 1 - \frac{1}{n}$ is Cauchy, but x_n does not converge.

* $(\mathbb{Q}, d_2)$ is <u>not</u> complete, because

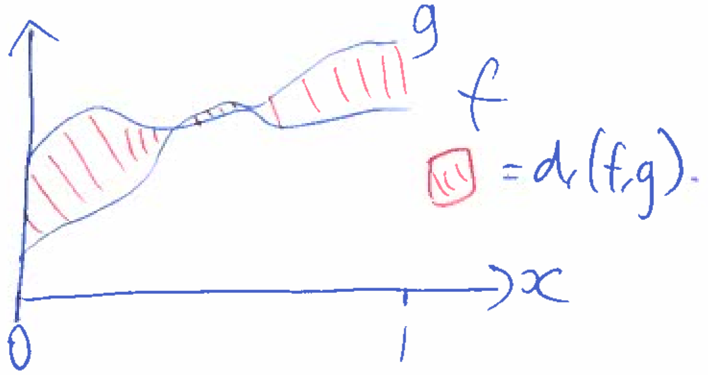
← rational numbers

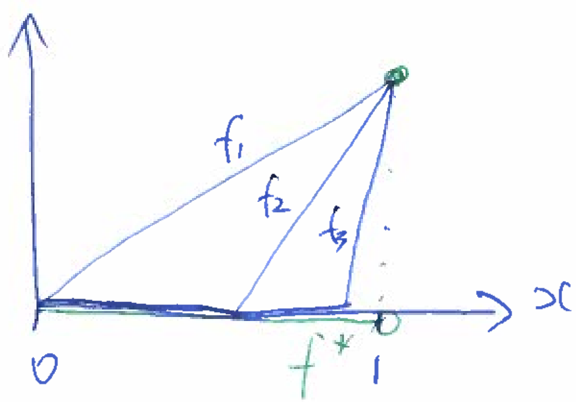
$x_1 = 3$, $x_2 = 3.1$, $x_3 = 3.14$, etc.

is Cauchy, but does not converge.

* $(\mathbb{R}, d_2)$ is complete. (Details later)

* $(CB([0,1]), d_1)$ where $d_1(f,g) = \int_0^1 |f(x) - g(x)|\,dx$ is <u>not</u> complete.





<u>Theorem C.9</u> Let (X, d) be any metric space. If $x_n \in X$ is a convergent sequence, then x_n is a Cauchy sequence.

<u>Proof</u> Suppose $x_n \to x^*$. Fix any radius $r > 0$. There is some N such that

$$d(x_n, x^*) < \frac{r}{2} \text{ for all } n > N.$$

By the triangle inequality,

$$d(x_n, x_m) \leq d(x_n, x^*) + d(x^*, x_m) < \frac{r}{2} + \frac{r}{2} = r$$

for all ~~x_n, x_m~~ $n,m > N$. So x_n is a Cauchy sequence. □

<u>Theorem C.10</u> Let (X,d) be any metric space. If $x_n \in X$ is a Cauchy sequence, and y_n is a subsequence of x_n, and $y_n \to y^*$, then $x_n \to y^*$.

<u>Theorem C.11</u> Let (X,d) be a metric space. If $x_n \in X$ is a Cauchy sequence, then x_n is bounded.

<u>Theorem C.12</u> Let (X,d) be a metric space. If $x_n \in X$ is a Cauchy sequence, and y_n is a subsequence of x_n, then y_n is a Cauchy sequence.

<u>Theorem C.13</u> $(\mathbb{R}, d_2)$ is a complete metric space.

$$\pi(p^g, p^d; p^w) = \max_{w \in \mathbb{R}_+} p^g g(w) + p^d d(w) - p^w w.$$

Eg 2.3

x crude input

$e = f(x)$ ethylene output

$y = g(e)$ plastic output

p^x, p^y prices of crude & plastic

$$\pi(p^y; p^x) = \max_{x \in \mathbb{R}_+} p^y g(f(x)) - p^x x$$

We can use first-order conditions to study optimal choices. In eg 2.3,

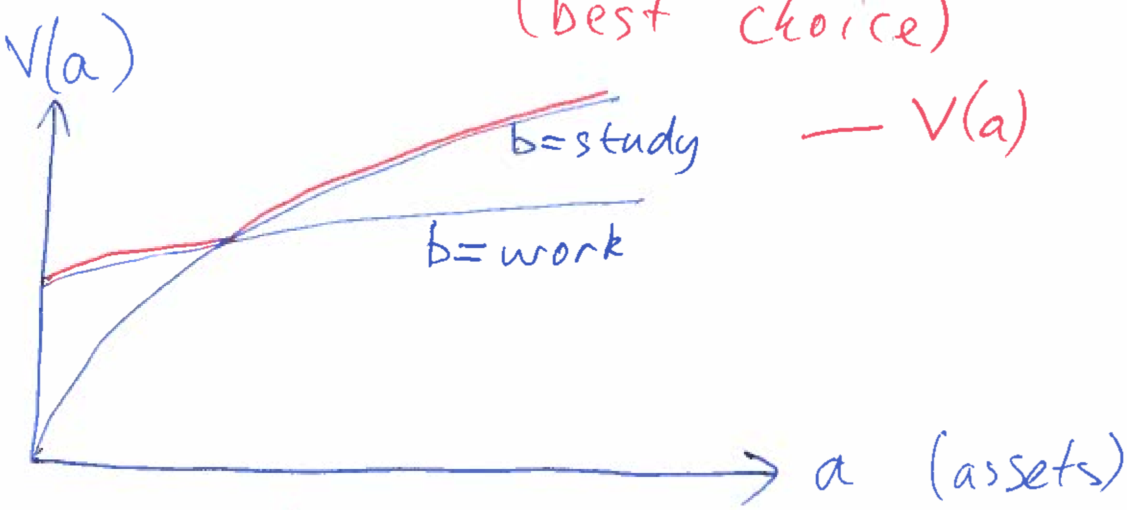
FOC x: $p^y g'(f(x)) f'(x) - p^x = 0$

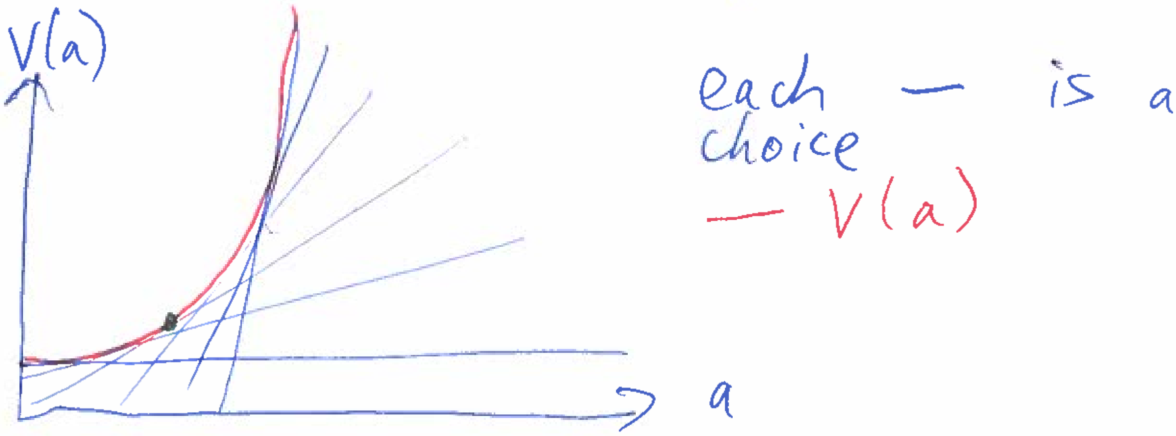
2.3 Upper Envelopes

How can we differentiate

$$\underbrace{V(\underbrace{a}_{\text{state variable}})}_{\text{value function}} = \max_{\underbrace{b}_{\text{choice variable}}} \underbrace{v(a, b)}_{\text{objective function}}$$

$$= v(a, \underbrace{b(a)}_{\text{policy function (best choice)}})$$





Theorem 2.1 (Envelope Theorem)

Let $v: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a differentiable function, $V(a) = \max_{b \in \mathbb{R}^m} v(a, b)$, and let $b(a)$ be its policy function. If V is differentiable function, then

$$V'(a) = \frac{\partial v(a, b)}{\partial a}\Big|_{b=b(a)}.$$

I didn't write

$$\frac{\partial v(a, b(a))}{\partial a}$$