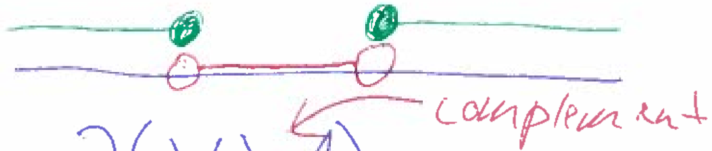
__Theorem C.6__ Let A be any subset of the metric space (X, d). Then A is an open set if and only if $X \setminus A$ is a closed set.

__Proof__



Notice that $\partial A = \partial(X \setminus A)$, because the two criteria for $x \in \partial A$:

① some $a_n \in A$ has $a_n \to x$,

② some $b_n \notin A$ has $b_n \to x$.

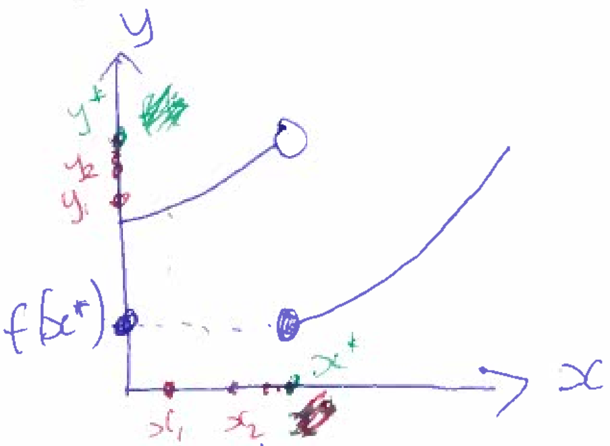
$\Rightarrow$ If A is open, then by Theorem C.5, A contains none of its boundary ∂A. So $\partial A \subseteq X \setminus A$ and $\partial(X \setminus A) = \partial A \subseteq X \setminus A$. By Theorem C.4, $X \setminus A$ is closed.

forward direction

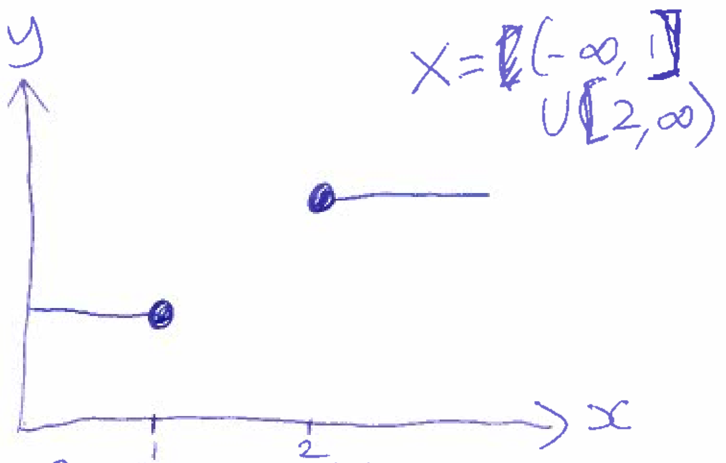
$\Leftarrow$ If $X \setminus A$ is closed, then by Theorem C.4, $\partial(X \setminus A) \subseteq X \setminus A$. Since $\partial(X \setminus A) = \partial A$, we know $\partial A \subseteq X \setminus A$. So A contains none of ∂A. $\square$

C.6 Continuity

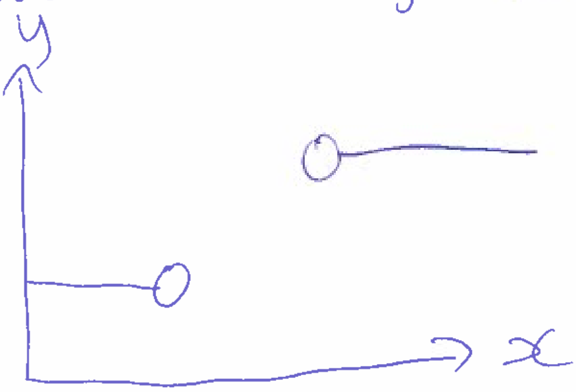
Def Consider two metric spaces (X, d_X) and (Y, d_Y). We say $f: X \to Y$ is <u>continuous at $x^* \in X$</u> if for every sequence of pre-images $x_n \in X$ with $x_n \to x^*$, the corresponding sequence (of images) $y_n = f(x_n) \in Y$ converges with $y_n \to f(x^*)$. We say that f is <u>continuous</u> if f is continuous ~~for~~ at all $x \in X$.



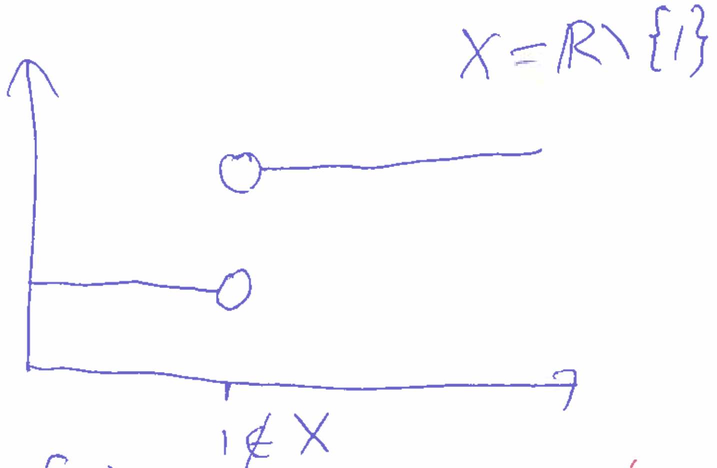
f is discontinuous at x^* since $y^* \neq f(x^*)$



f is continuous



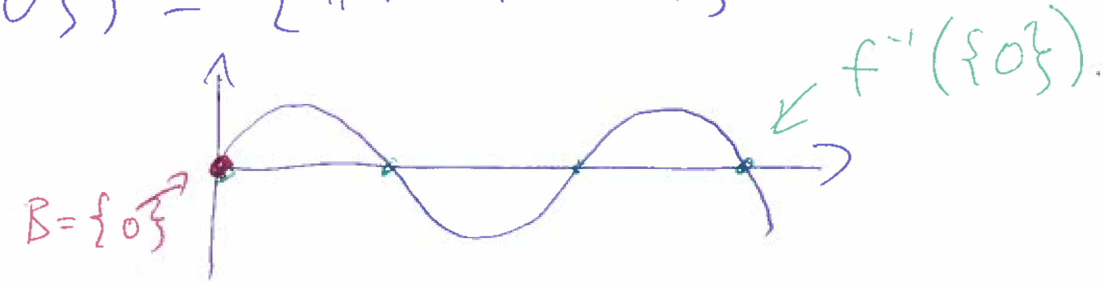
f is continuous



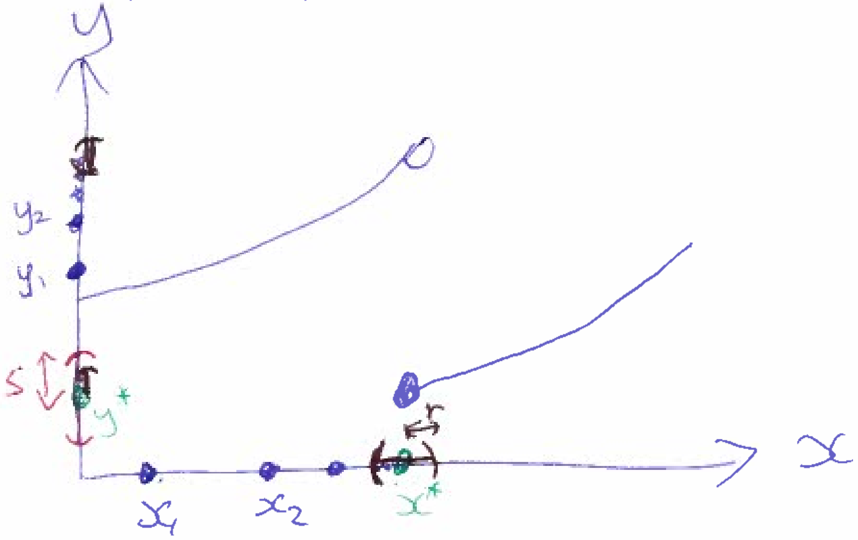
f is continuous not discontinuous at 1, since $1 \notin X$.

Def If $f: X \to Y$, then the *image* of $A \subseteq X$ is $f(A) = \{f(a) : a \in A\}$, and the *pre-image* of $B \subseteq Y$ is $f^{-1}(B) = \{x \in X : f(x) \in B\}$.

eg. consider $f(x) = \sin x$.
$f^{-1}(\{0\}) = \{\pi n : n \in \mathbb{Z}\}$.



Theorem C.7 Let $f: X \to Y$ be a function from (X, d_X) to (Y, d_Y). Pick any $x^* \in X$ and let $y^* = f(x^*)$. Then f is continuous at x^* if and only if for every open ball $B_s(y^*)$, there exists some open ball $B_r(x^*)$ such that $f(B_r(x^*)) \subseteq B_s(y^*)$.



<u>Proof</u> We will prove the contrapositive, i.e. that if the sequence def fails, then the open ball def fails, and vice versa.

Suppose that for some sequence $x_n \to x^*$ for which $y_n = f(x_n) \not\to y^*$. We will find an open ball $B_s(y^*)$ such that every open ball has $f(B_r(x^*)) \not\subseteq B_s(y^*)$. Since $y_n \not\to y^*$, there is some $s>0$ such that no tail of y_n lies inside $B_s(y^*)$. Since every open ball $B_r(x^*)$ contains a tail of x_n, it follows that for all $r>0$, $f(B_r(x^*)) \not\subseteq B_s(y^*)$.

Conversely, suppose that for some open ball $B_s(y^*)$, there is no ball $B_r(x^*)$ such that $f(B_r(x^*)) \subseteq B_s(y^*)$. We will construct a sequence $x_n \to x^*$ s.t. $f(x_n) \not\to y^*$. For every n, there exists some $x_n \in B_{\frac{1}{n}}(x^*)$ such that $f(x_n) \notin B_s(y^*)$. Therefore $x_n \to x^*$ but $f(x_n) \not\to y^*$. $\square$

<u>Theorem C.8</u> Let $f: X \to Y$ be a function from (X, d_X) to (Y, d_Y). Then f is continuous if and only if $f^{-1}(U)$ is an open set for all open sets $U \subseteq Y$.

2.2 Profit Maximisation

$$\underbrace{w \cdot x}_{\sum_{n=1}^{N-1} w_n x_n}$$

$w \in \mathbb{R}_+^{N-1}$ factor prices

$$\underbrace{\pi(p; w)}_{\text{profit function}} = \max_{x \in \mathbb{R}_+^{N-1}} \underbrace{p f(x)}_{\text{revenue}} - \underbrace{w \cdot x}_{\text{cost}}$$

$p \in \mathbb{R}_+$ sale price

$$= p\, f(x(p;w)) - w \cdot x(p;w)$$

$\boxed{\text{the}}$? policy function, factor demand function

FOC's x_i: $p \dfrac{\partial f(x)}{\partial x_i} - w_i = 0.$

Eg 2.1

r royalties (price of music)

l_m musician input (hours)

l_t technician input

w_m, w_t wages of musicians & technicians

$f(l_m, l_t)$ music output

$$\pi(r; w_m, w_t) = \max_{l_m, l_t} r f(l_m, l_t) - w_m l_m - w_t$$

Eg 2.2

w waste input

$g(w)$ glycerine output

$d(w)$ diesel output

p^w, p^g, p^d prices of waste, glycerine, diesel

$$\pi(p^g, p^d; p^w) = \max_{w \in \mathbb{R}_+} p^g g(w) + p^d d(w) - p^w w.$$

Eg 2.3

x crude input

$e = f(x)$ ethylene output

$y = g(e)$ plastic output

p^x, p^y prices of crude & plastic

$$\pi(p^y; p^x) = \max_{x \in \mathbb{R}_+} p^y g(f(x)) - p^x x.$$