Unofficial volunteer

Jingyi Li s1602211@sms.ed.ac.uk

C.4 Closed Sets

Def Consider a set A in the metric space (X, d). We say A is closed if there is no sequence $a_n \in A$ such that $a_n \to a^*$ and $a^* \notin A$.

Eg: crazy restaurant, the menu $A = [0, 1)$ inside $(\mathbb{R}_+, d_2)$ is not closed because $0.9, 0.99, 0.999, \ldots \to 1 \notin A$.

Theorem Suppose A is a set inside (X, d). Then A is closed if and only if A contains its boundary, i.e. $\partial A \subseteq A$.

Proof closed $\Rightarrow$ contains boundary:

Pick $x \in \partial A$. We want to prove $x \in A$.

Since $x \in \partial A$, there exists a sequence $a_n \in A$ s.t. $a_n \to x$. Since A is closed, $x \in A$.

contains boundary $\Rightarrow$ closed:

Assume for the sake of contradiction that A is not closed (but A does contain its boundary). Specifically, there is a sequence $a_n \in A$ s.t. $a_n \to x$ and $x \notin A$.

Let $b_n = x$. Notice that $b_n \to x$.

So we deduce $x \in \partial A$. Since $\partial A \subseteq A$, we conclude $x \in A$. ↯

More examples:

* $[0,1]$ is closed in $(\mathbb{R}, d_2)$.
* If (X,d) is a metric space, then X and $\emptyset$ are closed.
* $(0,1)$ is closed in $((0,1), d_2)$
* $(0,1)$ is NOT closed in $(\mathbb{R}, d_2)$.

Def Let A be a set in (X,d).

The closure of A is

$$cl(A) = \{x^* \in X : \text{there is a sequence } x_n \in A \text{ with } x_n \to x^*\}.$$

C.5 Open Sets

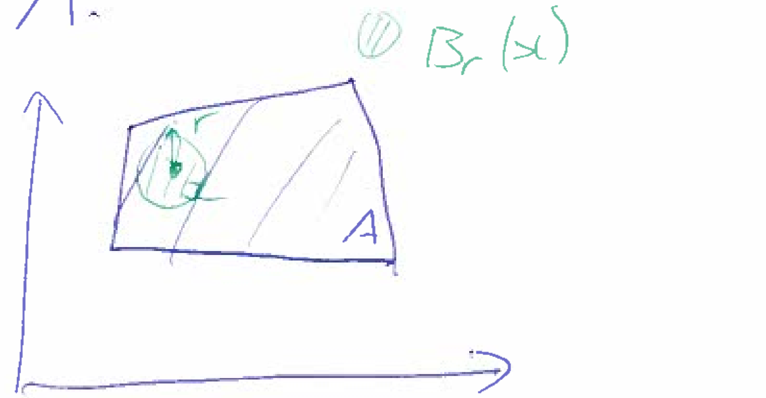
Def Let A be a set in (X,d).

We say $x \in A$ is an *interior point* of A if there is an ball $B_r(x)$ such that $B_r(x) \subseteq A$.

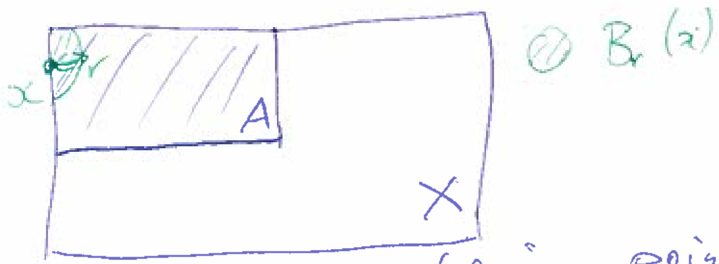
We say A is an *open set* if every point $a \in A$ is an interior point.

We say the set of interior points of A is the *interior*.

If $x \in A$ and A open set, we say that A is *open neighbourhood* of x.



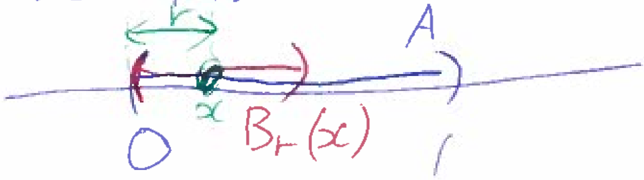
x is an interior point



x is an interior point

x is NOT an interior point

Eg: $(0,1)$ inside $(\mathbb{R}, d_2)$ is open set because if $x \in (0,1)$, pick $r = \min\{d(0,x), d(1,x)\}$ gives $B_r(x) \subseteq (0,1)$.



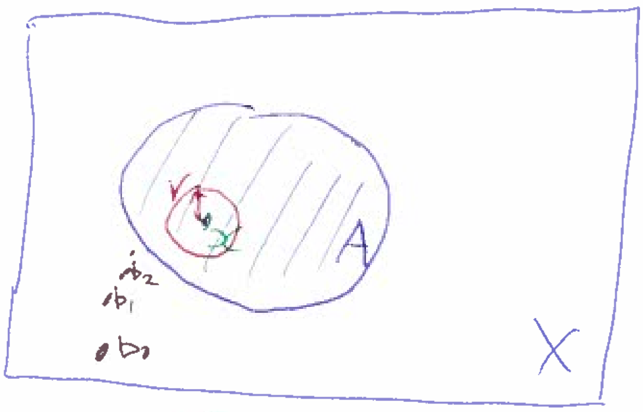
* Any $B_r(x)$ is an open set in (X,d).

* If (X,d) is a metric space, then X and $\emptyset$ are open sets.
* $[0,1)$ is neither open nor closed inside $(\mathbb{R}, d_2)$.
* $[0,1]$ is open in $([0,1], d_2)$ but not in $(\mathbb{R}, d_2)$.

Theorem Let A be a subset of a metric space (X,d). Then A is open if and only if A does not contain any of its boundary, i.e. $A \cap \partial A = \emptyset$.

Proof open $\Rightarrow$ contains none of boundary:

Pick any point $x \in A$. We want to prove $x \notin \partial A$. Since A is open, x is an interior point, so there exists $B_r(x)$ such that $B_r(x) \subseteq A$. Therefore, no sequence outside of A can converge to x. So $x \notin \partial A$.



<u>A is not open</u>

<u>$\Rightarrow$ A contains some of its boundary</u>

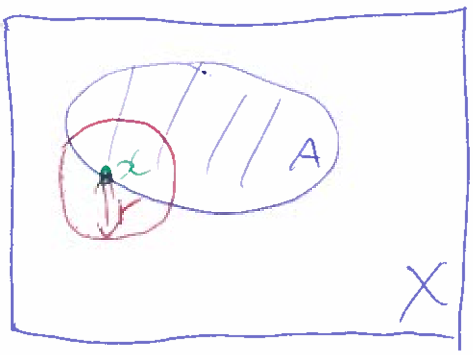
contrapositive of "contains none of boundary $\Rightarrow$ open"

Since A is not open, there is some point $x \in A$ such that for all $r > 0$, the ball $B_r(x) \not\subseteq A$. We will prove $x \in \partial A$, and hence A contains some of its boundary.

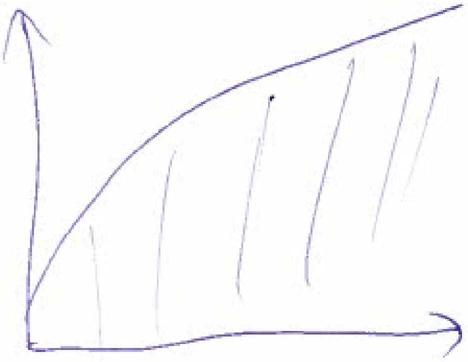
Let $a_n = x$. Then $a_n \in A$ and $a_n \to x$.

Let $r_n = \frac{1}{n}$, and pick $b_n \in B_{r_n}(x)$ but with $b_n \notin A$. Since $d(b_n, x) < \frac{1}{n}$, we know $b_n \to x$.

So we conclude that $x \in \partial A$. $\square$



D Convex Geometry cont'd



concave

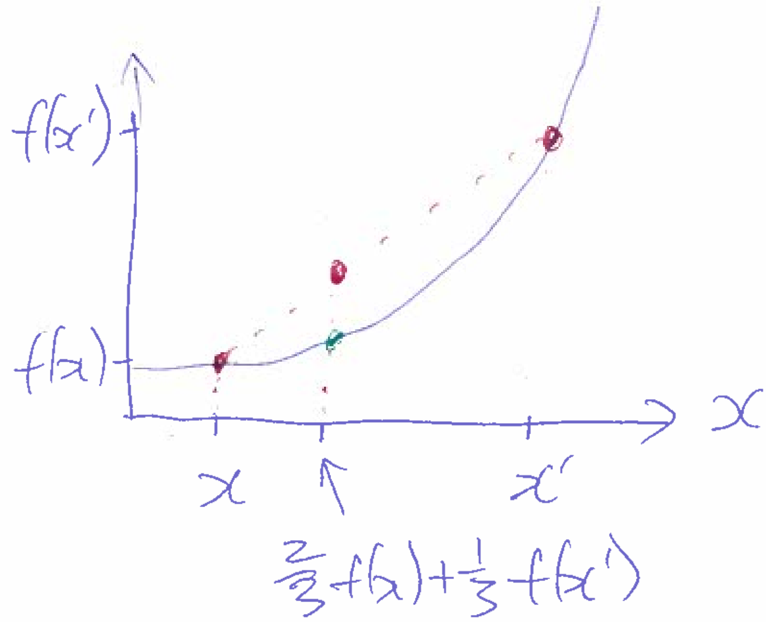
concave & convex

Theorem D.3 Suppose $f: \mathbb{R} \to \mathbb{R}$ is differentiable. Then f is a convex function if and only if its derivative f' is weakly increasing.

Theorem D.4 Suppose $f: \mathbb{R} \to \mathbb{R}$ is twice differentiable. Then f is convex iff (if and only if) $f''(x) \geq 0$ for all x.

Theorem D.5 A function $f: X \to \mathbb{R}$ is convex if and only if X is convex and for all $x, x' \in X$ and all $a \in (0,1)$,

$$\underbrace{a f(x) + (1-a) f(x')}_{\text{line}} \geq \underbrace{f(ax + (1-a)x')}_{\text{curve}}$$



Back to 2.1

We could assume:

* f is concave. Claim: $\Longrightarrow$

f has weakly decreasing marginal productivity and weakly decreasing returns to scale.