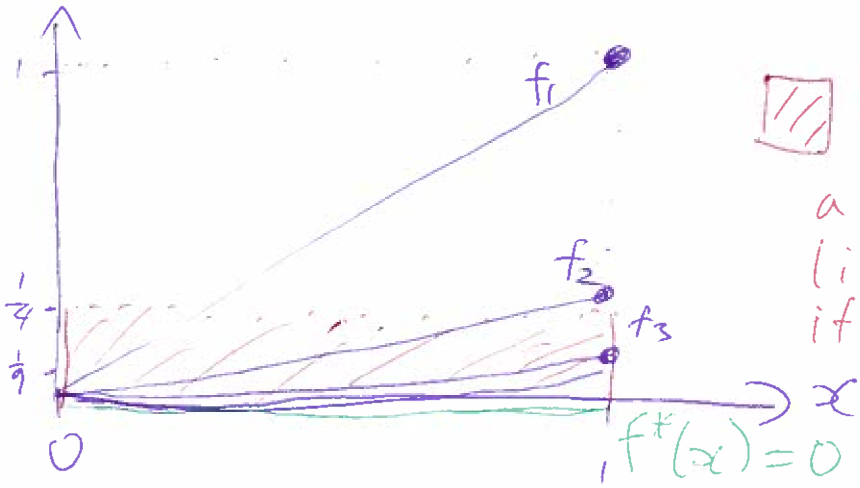
# C.2 Convergence (Cont'd)

Recall the sequence $f_n(x) = \frac{x}{n^2}$ inside the metric space $(B([0,1]), d_\infty)$.




▨ $B_{\frac{1}{4}}(f^*)$, i.e. a function $g$ lies inside $B_{\frac{1}{4}}(f^*)$ if and only if $g$ can be drawn inside the ▨ area.

**Claim:** $f_n \to f^*$.

**Proof:** Note that $d_\infty(f_n, f^*) = f_n(1)$.

We would like to show that for all $r > 0$, there exists $N$ s.t.

$$d_\infty(f_n, f^*) < r \quad \text{for all } n > N,$$

$$\Leftrightarrow \quad f_n(1) < r \quad \text{for all } n > N,$$

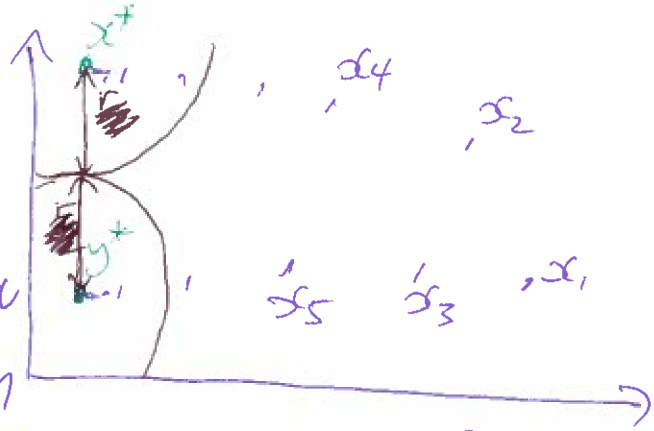
$$\Leftrightarrow \quad \frac{1}{n^2} < r \quad \text{for all } n > N.$$

So if we pick $N = \left\lceil \frac{1}{\sqrt{r}} \right\rceil$ (rounding up), then it will be true that

$$d_\infty(f_n, f^*) < r \quad \text{for all } n > N. \qquad \square$$

**Theorem C2** A sequence $x_n$ in $(X, d)$ can converge to at most one point.

**Proof** Suppose for the sake of contradiction that $x_n$ converges to two points, $x_n \to x^*$ and $y^*$ where $x^* \neq y^*$.




Let $r = \frac{1}{2} d(x^*, y^*)$.

Since $x_n \to x^*$, there exists some $N$ s.t. $d(x_n, x^*) < r$ for all $n > N$ and similarly, some $M$ s.t. $d(x_n, y^*) < r$ for all $n > M$.

Let $S = \max\{N, M\} + 1$. Then $d(x_S, x^*) < r$ and $d(x_S, y^*) < r$.

By the triangle inequality,

$$\begin{aligned} d(x^*, y^*) &\leq d(x^*, x_S) + d(x_S, y^*) \\ &< r + r \\ &= \tfrac{1}{2} d(x^*, y^*) + \tfrac{1}{2} d(x^*, y^*) \\ &= d(x^*, y^*). \end{aligned}$$

$\lightning$

**Def** We say that $y_n$ is a **subsequence** of $x_n$ if there exists a strictly increasing sequence $k_n \in \mathbb{N}$ (i.e. $k_{n+1} > k_n$) such that $y_n = x_{k_n}$.

**Theorem C3** If $x_n \to x^*$ and $y_n$ is a subsequence $x_n$ then $y_n \to x^*$.

**Proof** Since $y_n$ is a subsequence, $y_n = x_{k_n}$ for some sequence $k_n$.

Pick any $r>0$. Since $x_n \to x^*$, there exists some $N$ s.t.

$$d(x_n, x^*) < r \quad \text{for all } n > N,$$

$$\Rightarrow d(x_{k_n}, x^*) < r \quad \text{for all } n > N$$

(since $k_n \geq n$)

$$\Rightarrow d(y_n, x^*) < r \quad \text{for all } n > N. \quad \square$$

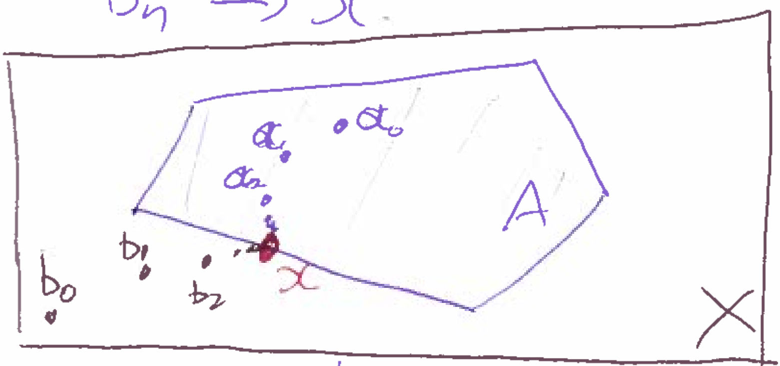
# C3 Boundaries

i.e. $A \subseteq X$

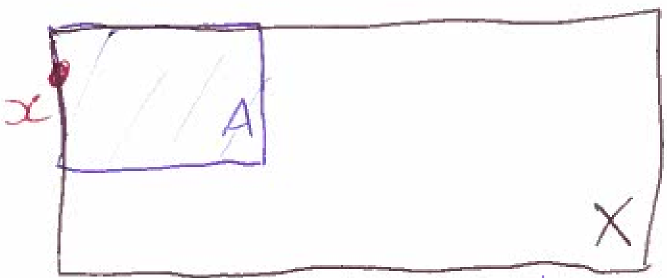
**Def** Let $A$ be a set inside $(X,d)$.
A point $x \in X$ is a <u>boundary point</u> of $A$ if

(i) there exists a sequence $a_n \in A$ s.t. $a_n \to x$, and

(ii) there exists a sequence $b_n \in \underbrace{X \setminus A}_{\text{outside of } A}$ s.t. $b_n \to x$.




$x$ is a boundary point of $A$.




$x$ is NOT a boundary point.

**Def** The <u>boundary</u> of $A$ is the set of boundary points of $A$, denoted $\partial A$.

More examples:

* In $(\mathbb{R}, d_2)$, $\partial[0,1] = \{0,1\}$.
* In $(\mathbb{R}, d_2)$, $\partial(0,1) = \{0,1\}$
* In $([0,1], d_2)$, $\partial[0,1] = \emptyset$.
* In $([0,1], d_2)$, $\partial(0,1) = \{0,1\}$.

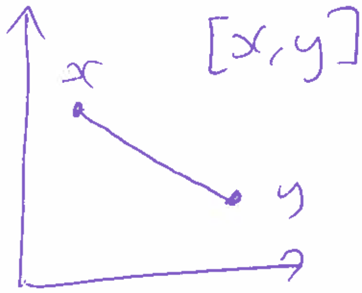
  $0 \in \partial(0,1)$ because

  * $a_n = \frac{1}{n} \in (0,1)$ and $a_n \to 0$.
  * $b_n = 0 \notin (0,1)$ and $b_n \to 0$.
* In $(\mathbb{R}_+, d_2)$, $\partial[0,1] = \{1\}$.

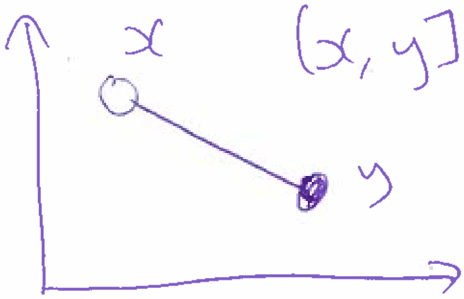
# D Convex Geometry

<u>Def</u> A <u>closed interval</u> between two points $x, y \in \mathbb{R}^n$ is defined as

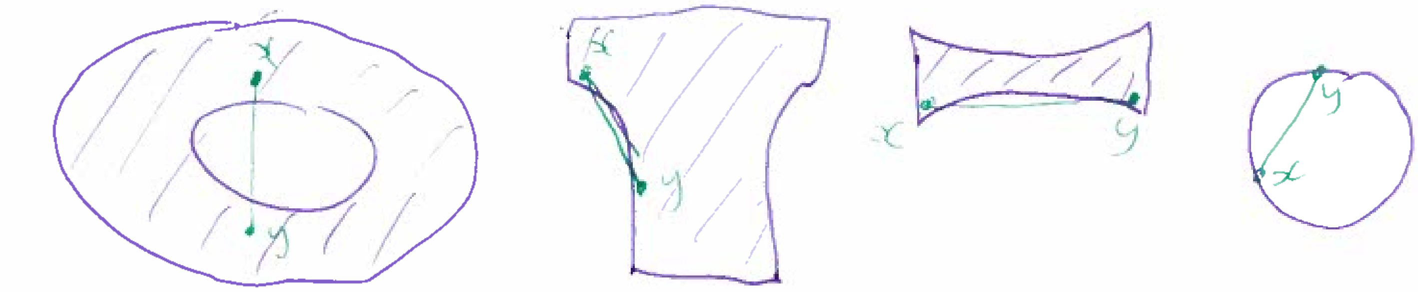
$$[x, y] = \{\underbrace{ax + (1-a)y}_{\substack{\text{convex}\\\text{combination}\\\text{or}\\\text{mixture}}} : a \in [0, 1]\}.$$



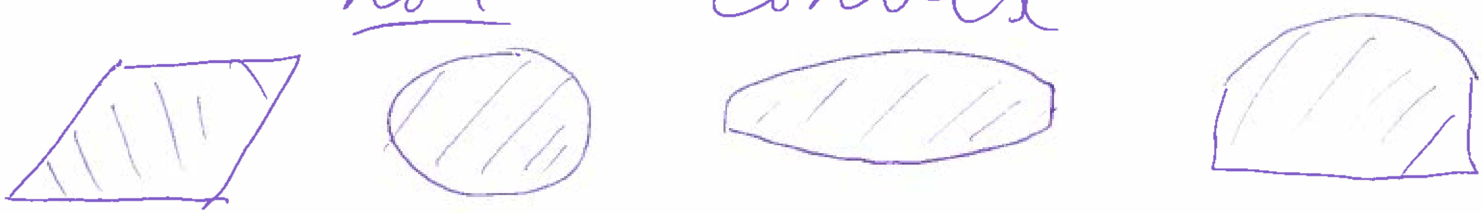

Similar definitions apply to $(x, y)$, $[x, y)$ and $(x, y]$.




<u>Def</u> $X \subseteq \mathbb{R}^n$ is <u>convex set</u> if for all $x, y \in X$, the interval $[x, y]$ is a subset of $X$.



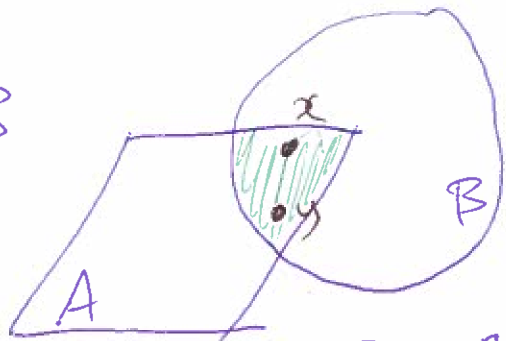

not convex



are convex

<u>Theorem</u> The intersection of convex sets is convex.

<u>Proof</u> Suppose $A$ and $B$ are convex sets.



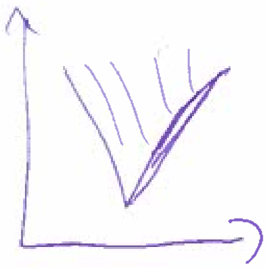

Pick any $x, y \in A \cap B$.
We would like to prove that $[x, y] \subseteq A \cap B$.
Since $x, y \in A \cap B$, we know $x, y \in A$.
Since $A$ is convex, $[x, y] \subseteq A$.
By a similarly line of reasoning, $[x, y] \subseteq B$. So $[x, y] \subseteq A \cap B$. $\square$

<u>Def</u> $f : X \to \mathbb{R}$ is a <u>convex function</u> if its <u>hypergraph</u>

$$\text{hyper}(f) = \{(x, y) : x \in X, y \geq f(x)\}$$

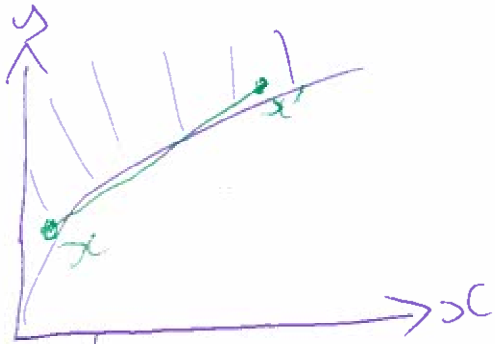
is a convex set.


f is a convex function



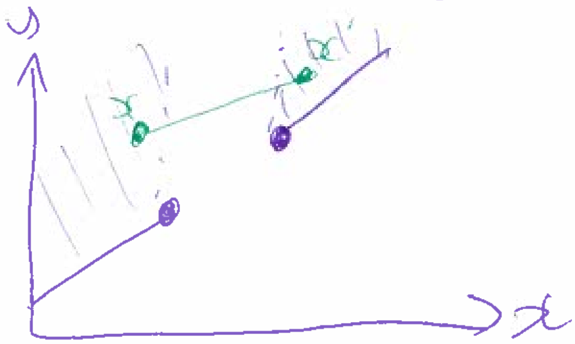
convex

convex

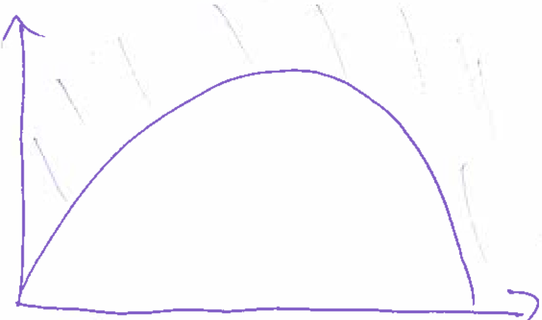



NOT convex


NOT convex




NOT convex

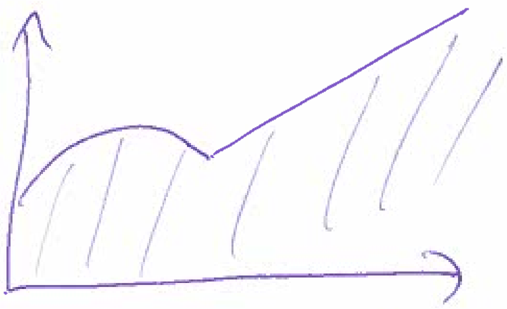


NOT convex

<u>Def</u> $f: X \to \mathbb{R}$ is a <u>concave function</u> if its <u>hypograph</u>

$$\text{hypo}(f) = \{(x,y) : x \in X,\ y \leq f(x)\}$$

is a convex set.



neither concave nor convex