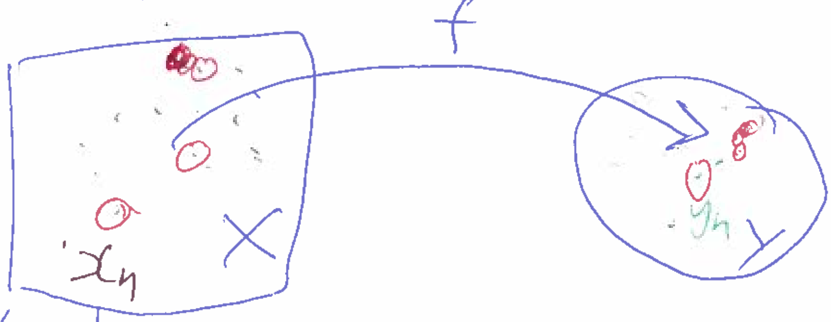
Theorem C.17 Suppose $f: X \to Y$ is a continuous function between (X, d_X) and (Y, d_Y). If X is compact and $Y = f(X)$ (i.e. f is surjective), then Y is compact.

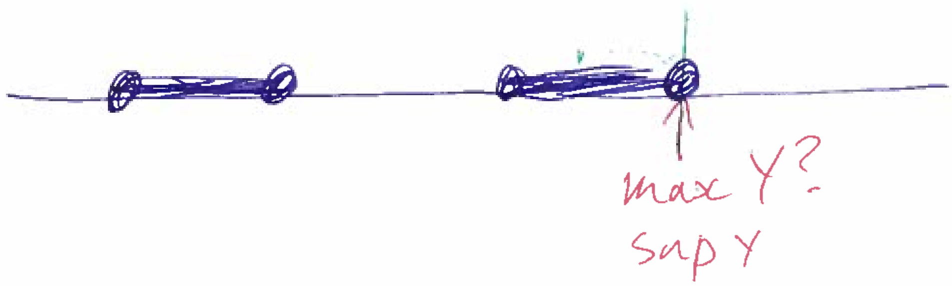


Proof Let $y_n \in Y$ be any sequence. Since $Y = f(X)$, there exists a a sequence $x_n \in X$ s.t. $y_n = f(x_n)$. Since X is compact, there is a convergent subsequence x_{n_k}. Since f is continuous, $y_{n_k} = f(x_{n_k})$ is convergent, and is a subsequence of y_n. $\square$

Theorem C.18 (Extreme Value Theorem) Suppose $f: X \to \mathbb{R}$ is a continuous function between (X, d) and $(\mathbb{R}, d_2)$. If X is compact and $X \neq \emptyset$, then f has maximum (and a minimum), i.e.

$$\max_{x \in X} f(x)$$ has a solution.

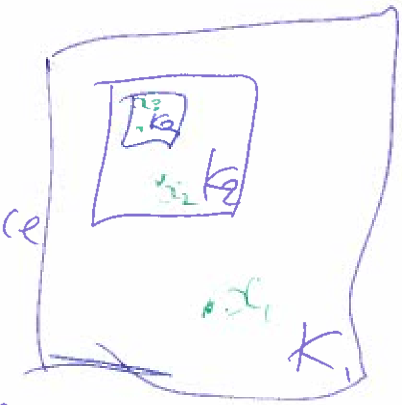
<u>Proof</u> Since (X, d) is compact and f is continuous and surjective on $(Y, d_2) = (f(X), d_2)$, Theorem C.17 implies (Y, d_2) is a compact metric space.
By the Bolzano-Weierstrass theorem, Y is closed and bounded inside $(\mathbb{R}, d_2)$.
Since Y is bounded, its supremum is finite. Since Y is closed, $\sup Y \in Y$. So Y has a maximum. $\square$



<u>Theorem C.21 (Cantor's intersection theorem)</u>
Let (X, d) be a metric space, and let K_n be a sequence of subsets of X. If each K_n is non-empty, compact, and nested $(K_{n+1} \subseteq K_n)$, then $\bigcap_{n=1} K_n \neq \emptyset$.

<u>Proof</u> Let x_n be any sequence with the property $x_n \in K_n$.

Since $x_n \in K_1$ and K_1 is compact, there is a convergent subsequence $y_n = x_{k_n}$. Also note that $y_n \in K_n$ since $K_{k_n} \subseteq K_n$.



Let $y^* = \lim_{n \to \infty} y_n$. Since K_1 is closed and $y_n \in K_1$, we know $y^* \in K_1$. Since K_2 is closed, and $y_2, y_3, \ldots \in K_2$, we know $y^* \in K_2$. Similarly $y^* \in K_n$.

So $y^* \in \bigcap_{n=1}^{\infty} K_n$. $\square$

To see ~~than~~ how this applies to Game theory, see Q29 Bvii.

C.11 Extreme Punishments

$$\min_{s \in \mathbb{R}_+,\, p \in [0,1]} c(p)$$

$$\text{s.t. } h \geq \underbrace{-ps + (1-p)b}_{\text{payoff to crime}}$$

s sanction

p probability of catching criminals (policing)

$c(p)$ cost of police force, increasing continuous

h honest payoff

b bounty payoff to criminals.

Let $f : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ be $f(s,p) = -ps + (1-p)b$.

The set of regimes that deter crime is $D = f^{-1}((-\infty, h])$.

Since $(-\infty, h]$ is a closed subset of $(\mathbb{R}, d_2)$, and f is continuous, we know D is closed inside $(\underbrace{\mathbb{R}_+ \times [0,1]}_{\text{set of all regimes}}, d_2)$.

Even though D is closed, it is not compact. To see this, consider $(s_n, p_n) = (n(b-h), \frac{1}{n})$.

Notice that $f(s_n, p_n) = h - \frac{b}{n} < h$.

But s_n diverges to ∞. So D is unbounded, and hence not compact.

So we can't use the Extreme Value Theorem.

Notice that welfare is $c(p_n) = c(\frac{1}{n}) \to c(0)$, which is the best possible welfare. But $p=0$ is not feasible. So $c(0)$ is unattainable.

e.g. $u(x) = \log x$.

$$F(\hat{V})(k) = \sup_{x, k' \geq 0} \log x + \beta \hat{V}(k')$$
$$\text{s.t. } x + k' = k$$

E.g. $\cancel{V_A(k) = \log x}$

$$F(u)(k) = \sup_{x, k' \geq 0} \log x + \beta \log k'$$
$$\text{s.t. } x + k' = k$$

~~FOC x~~ $$= \sup_{x \in [0,k]} \log x + \beta \log(k - x)$$

FOC x: $$\frac{1}{x} + \frac{\beta}{k-x}(-1) = 0$$

$$\Leftrightarrow \quad \frac{1}{x} = \frac{\beta}{k-x}$$

$$\Leftrightarrow \quad x = \frac{k-x}{\beta}$$

$$\Leftrightarrow \quad x\left(1 + \frac{1}{\beta}\right) = \frac{k}{\beta}$$

$$\Leftrightarrow \quad x(\beta+1) = k$$

$$\Leftrightarrow \quad x = \frac{k}{\beta+1}.$$

So $$F(u)(k) = \log\left(\frac{k}{\beta+1}\right) + \beta \log\left(k - \frac{k}{\beta+1}\right)$$

Q 3(B)(i): Prove $A=\overline{B}_r(x) = \{y \in X : d(x,y) \le r\}$

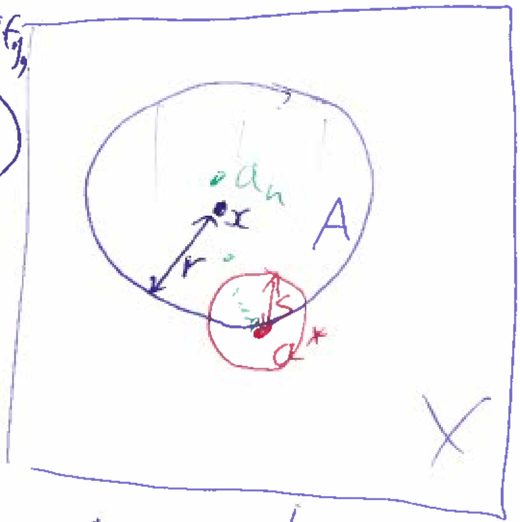
is a closed set in (X,d).

not <

A: Let $a_n \in A$ be a convergent sequence with $a_n \to a^*$. We would like to prove $a^* \in A$.

Since $a_n \in A$, we know $d(a_n, x) \le r$.

By the triangle inequality,



$$d(a^*, x) \le d(x, a_n) + d(a_n, a^*)$$
$$\le r + d(a_n, a^*)$$
for all n.

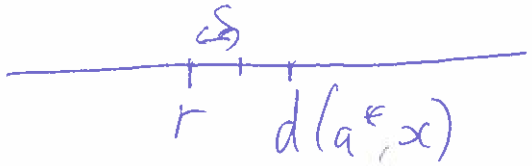
Since $a_n \to a^*$, for every $s>0$, there exists some N s.t.

$$d(a_n, a^*) < s \quad \text{for all } n > N.$$

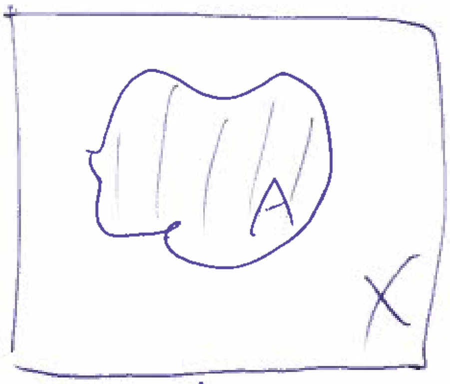
Combining, for all $s>0$,

$$d(a^*, x) \le r + s.$$

So $d(a^*, x) \le r$, and $a^* \in A$. $\square$



is (X, d_X) compact?



Is A compact inside (X, d_X)?