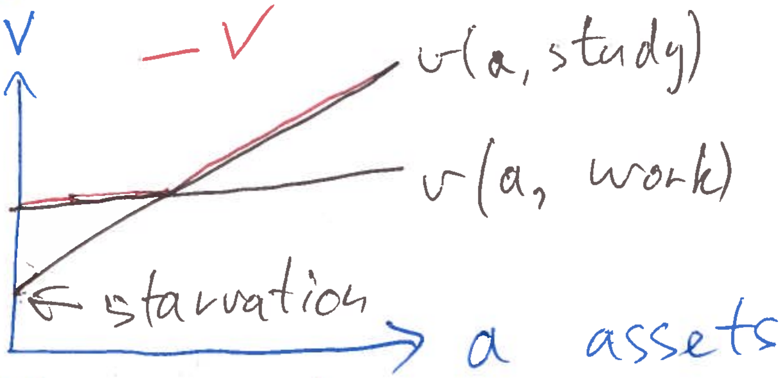
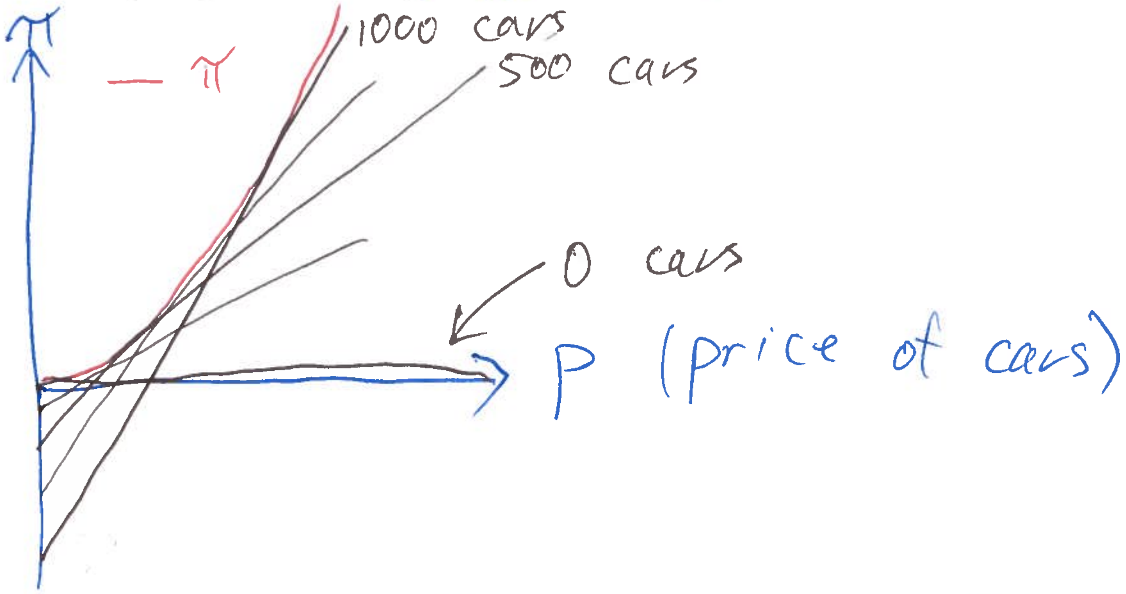
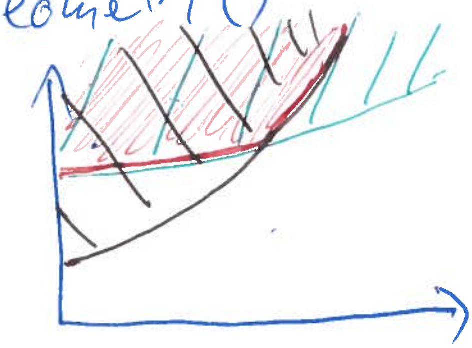
Section 2.3 – cont'd

Theorem 2.2 Suppose V is the upper envelope of convex functions, i.e. $V(a) = \max_b v(a,b)$ where $v(\cdot, b)$ is a convex function for each b. Then V is convex.





Proof 1 (Geometric)



want to prove 🟥 is convex

$=$ ⬛ $\cap$ 🟩 is convex

We want to prove that hyper(V) is a convex set. Since ~~each~~ hyper(v(·,b)) is a convex set for all b, and

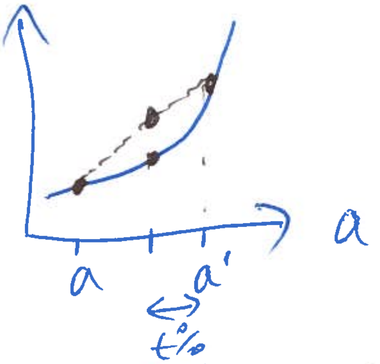
$$\text{hyper}(V) = \bigcap_{b} \text{hyper}(v(\cdot, b))$$

and intersections of convex sets are convex, we conclude hyper(V) is convex. □

Proof 2

We want to prove that for all a and a', and all $t \in [0,1]$,

$$\underbrace{tV(a) + (1-t)V(a')}_{\text{line}} \geq \underbrace{V(ta+(1-t)a')}_{\text{curve}}.$$



Starting with LHS,

$$tV(a) + (1-t)V(a')$$

$$= t\,v(a, \overset{\text{policy function}}{\boxed{b(a)}}) + (1-t)\,v(a', b(a'))$$

$$\geq t\,v(a, \underset{\text{worse choice}}{b(ta+(1-t)a')}) + (1-t)\,v(a', b(a'))$$

$$\geq t\, v(a, b(ta+(1-t)a')) + \underbrace{(1-t)v(a', b(ta+(1-t)a'))}_{\text{worse choice}}$$

$$\geq v(ta + (1-t)a', b(ta+(1-t)a'))$$

[since $v(\cdot, b)$ for any b (including this particular one)

$$= V(ta + (1-t)a'). \qquad \square$$

Theorem 2.3 For ~~pr~~ every production function f, the firm's profit function π is a convex function. Hence if π is smooth, then $\frac{\partial y(p;w)}{\partial p} \geq 0$ and $\frac{\partial x_i(p;w)}{\partial w_i} \leq 0$.

Proof Recall $\pi(p;w) = \max_{x \in \mathbb{R}^{N-1}_+} pf(x) - w \cdot x$.

We can write the objective as

$$v(\underbrace{p, w}_{a}; \underbrace{x}_{b}) = pf(x) - w \cdot x,$$

which is linear (and hence convex) in prices (p,w). So Theorem 2.2 implies π is a convex function.

Recall that by the envelope theorem,

$$\frac{\partial \pi(p;w)}{\partial p} = \overbrace{y(p;w)}^{\text{supply}} \quad \text{and} \quad \frac{\partial \pi(p;w)}{\partial w_i} = -x_i(p;w)$$

Since π is convex, $\frac{\partial \pi(p;w)}{\partial p}$ is increasing in p and $\frac{\partial \pi(p;w)}{\partial w_i}$ is increasing in w_i. Since LHS's are increasing (in the relevant price, p ~~and~~ or w_i), so is the RHS.

We conclude $y(p;w)$ is increasing in p and $x_i(p;w)$ is decreasing in w_i. $\square$

2.4 Dynamic Programming and Cost Production

Recall $\pi(p;w) = \max_{x \in \mathbb{R}_+^{N-1}} pf(x) - w \cdot x$.

Let's focus on the output choice y.

We would like to write

$$\pi(p;w) = \max_{y \in \mathbb{R}_+} py - c(y;w)$$

↖ Bellman equation

where $$\underbrace{c(y;w)}_{\text{cost function}} = \min_{x\in\mathbb{R}^{N-1}_+} w\cdot x \quad \text{s.t. } \underbrace{f(x)}_{\text{output}} \geq \underbrace{y}_{\text{production target}}.$$

Lemma (Principle of Optimality)

The Bellman equation holds.

Proof

$$\max_{x\in\mathbb{R}^{N-1}_+} p\,f(x) - w\cdot x$$

$$= \max_{y\in\mathbb{R}_+,\ x\in\mathbb{R}^{N-1}_+} pf(x) - w\cdot x \quad \text{s.t. } f(x) = y \leftarrow \text{or } \geq, \text{ as long as } f \text{ is increasing}$$

$$= \max_{y\in\mathbb{R}_+} \left[\max_{x\in\mathbb{R}^{N-1}_+} pf(x) - w\cdot x \quad \text{s.t. } f(x)=y\right]$$

recall 2nd homework

$$= \max_{y\in\mathbb{R}_+} \left[\max_{x\in\mathbb{R}^{N-1}_+} p\,\textcircled{y} - w\cdot x \quad \text{s.t. } f(x)=y\right]$$

$$= \max_{y\in\mathbb{R}_+} \left\{py + \left[\max_{x\in\mathbb{R}^{N-1}_+} - w\cdot x \quad \text{s.t. } f(x)=y\right]\right\}$$

$$= \max_{y\in\mathbb{R}_+} \left\{ py - \underbrace{\begin{bmatrix} \displaystyle\min_{x\in\mathbb{R}_+^{N-1}} & w\cdot x \\ \text{s.t.} & f(x)=y \end{bmatrix}}_{\parallel\ c(y;w)} \right\}$$

$$= \max_{y\in\mathbb{R}_+} py - c(y;w).$$

□

Theorem 2.4 $p = \left.\dfrac{\partial c(y;w)}{\partial y}\right|_{y=y(p;w)}.$

Proof Since the Bellman equation holds, we know the first-order condition of the equation is satisfied. □

3.2 Time Preference

Cake-eating problem: T time periods

$$V_t(k_t) = \max_{x_t, x_{t+1}, \dots, x_T} u_t(x_t) + u_{t+1}(x_{t+1}) + \cdots u_T(x_T)$$

$$\text{s.t. } x_t + x_{t+1} + \cdots + x_T = k$$

$\uparrow$ cake at start of t

$\nwarrow$ cake consumed today (t)

Bellman equation:

$$\boxed{V_t}(k_t) = \max_{x_t, k_{t+1}} u_t(x_t) + \boxed{V_{t+1}}(k_{t+1})$$

$$\text{s.t. } x_t + k_{t+1} = k_t.$$

Lemma (Principle of Optimality)

$$V_t(k_t) = \max_{x_t, \dots, x_T \geq 0} u_t(x_t) + \cdots + u_T(x_T)$$

$$\text{s.t. } x_t + \cdots + x_T = k_t$$

$$= \max_{\boxed{k_{t+1}}, x_t, \dots, x_T \geq 0} u_t(x_t) + \cdots + u_T(x_T)$$

$$\text{s.t. } x_t + \cdots + x_T = k_t,$$

$$\boxed{x_t + k_{t+1} = k_t.}$$

$$= \max_{\substack{x_t,\, k_{t+1} \geq 0 \\ \text{s.t. } x_t + k_{t+1} = k_t}} \left[\max_{\substack{x_{t+1}, \dots, x_T \\ \text{s.t. } x_t + \dots + x_T = k_t}} u_t(x_t) + \dots + u_T(x_T) \right]$$

$$\max_{x,y} f(x,y) = \max_{x} g(x) = \max_{x} \max_{y} f(x,y)$$

where $g(x) = \max_{y} f(x,y)$

$$= \max_{\substack{x_t,\, k_{t+1} \geq 0 \\ \text{s.t. } x_t + k_{t+1} = k_t}} \enclose{circle}{u_t(x_t)} + \left[\max_{\substack{x_{t+1}, \dots, x_T \\ \text{s.t. } x_t + \dots + x_T = k_t}} \enclose{circle}{u_{t+1}(x_{t+1}) +} \dots + u_T(x_T) \right]$$

$$= \max_{\substack{x_t,\, k_{t+1} \geq 0 \\ \text{s.t. } x_t + k_{t+1} = k_t}} u_t(x_t) + \underbrace{\left[\max_{\substack{x_{t+1}, \dots, x_T \\ \text{s.t. } \enclose{circle}{x_{t+1}} + \dots + x_T = \enclose{circle}{k_{t+1}}}} u_{t+1}(x_{t+1}) + \dots + u_T(x_T) \right]}_{V_{t+1}(k_{t+1})}$$

since $k_t - x_t = k_{t+1}$

$$= \max_{x_t,\, k_{t+1} \geq 0} u_t(x_t) + V_{t+1}(k_{t+1})$$
$$\text{s.t. } x_t + k_{t+1} = k_t. \qquad \square$$

Compare FOCs x_t:

In original formulation: $u_t'(x_t) - \lambda = 0$ ↙ Lagrange multiplier

In Bellman formulation: $u_t'(x_t) + V_{t+1}'(k_t - x_t)(-1) = 0$

$$\iff u_t'(x_t) = V_{t+1}'(k_t - x_t).$$

marginal utility today = marginal value of [illegible] saving for future.

Appendix G

In the finite horizon (T periods), there is a final period, whose value function is

$$V_T(k_T) = u_T(k_T).$$

In infinite horizon problems, there is no final period.

$$V_t(k) = \max_{\{x_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} u(x_s)$$

$$\text{s.t.} \sum_{s=t}^{\infty} x_s = k.$$

Note: we now have the same utility function $u(x)$ each period, except for discounting β^t.