C7 Completeness

$(X, d) = ((0,1], d_2)$ has a "hole" in it at 0.

Idea: $x_n = \frac{1}{n+1}$ "wants" to converge to 0, but $0 \notin X$ — there is a hole.

Def Let (X, d) be a metric space. A sequence $x_n \in X$ is called a Cauchy sequence "wants to converge" if for every radius $r > 0$, there exists an N such that for all $n, m > N$,

$$d(x_n, x_m) < r.$$

Def (X, d) is complete "no holes" if every Cauchy sequence $x_n \in X$ is convergent.

Examples:

* $(\mathbb{R}, d_2)$ is complete.
* $((0,1], d_2)$ is not complete.
* $(\mathbb{Q}, d_2)$ - rational numbers - is not complete

$x_1 = 3$

$x_2 = 3.1$

$x_3 = 3.14$

$x_4 = 3.141$

$x_5 = 3.1415$

$\vdots$

$x_n \to \pi$ in $(\mathbb{R}, d_2)$.

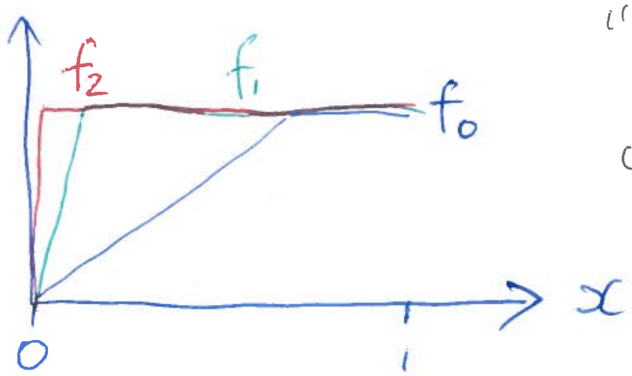
x_n is a Cauchy sequence inside $(\mathbb{Q}, d_2)$, but $\pi \notin \mathbb{Q}$, x_n does not converge.

$*\,(CB([0,1]), d_1)$

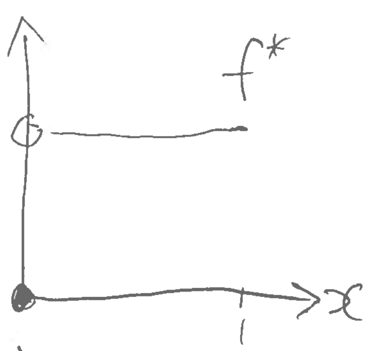
where $CB([0,1]) = \{ f : [0,1] \to \mathbb{R},$ f is continuous and bounded $\}$

and $d_1(f, g) = \int_0^1 |f(x) - g(x)|\, dx$.

This space is <u>not</u> complete.



"wants" to converge to



$d_1(f_n, f^*) \to 0$.

But $f^* \notin CB([0,1])$.

Theorem If $x_n \in X$ is a convergent sequence, then x_n is a Cauchy sequence.

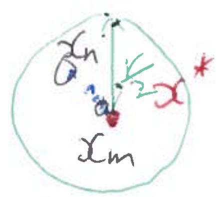
Proof Suppose $x_n \to x^*$. Fix any radius $r > 0$. Since $x_n \to x^*$, there exists some N such that

$$d(x_n, x^*) < \frac{r}{2} \quad \text{for all } n > N.$$

By the triangle inequality,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x^*) + d(x_m, x^*) \\ &< \frac{r}{2} + \frac{r}{2} \\ &= r \end{aligned}$$

for all $n, m > N$. $\square$



Theorem If $x_n \in X$ is a Cauchy sequence and $y_n \to y^*$ is a convergent subsequence of x_n, then $x_n \to y^*$.

__Proof__ Pick any $r>0$. Since x_n is a Cauchy sequence, there is some N such that

$$d(x_n, x_m) < \frac{r}{2} \quad (\ast\ast) \quad \text{for all } n,m>N.$$

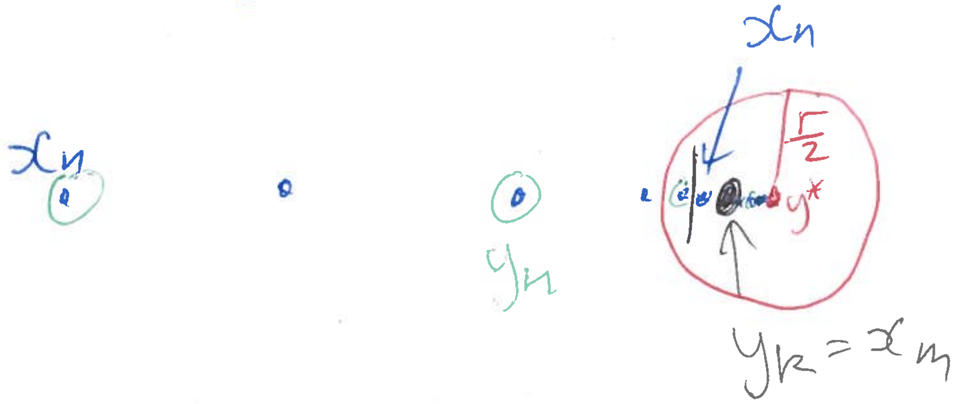
Since y_n is a convergent sequence, there is some $k>N$ such that

$$d(y_k, y^*) < \frac{r}{2}. \quad (\ast)$$

~~Pick~~ Since y_n is a subsequence of x_n, we can pick x_m such that $x_m = y_k$. By the triangle inequality,

$$d(x_n, y^*) \leq d(x_n, \overset{x_m}{y_k}) + d(y_k, y^*)$$
$$< \underset{(\ast\ast)}{\frac{r}{2}} + \underset{(\ast)}{\frac{r}{2}}$$
$$= r.$$

So $x_n \to y^*$. $\square$



Theorem If x_n is a Cauchy sequence, then x_n is bounded.

Theorem If x_n is a Cauchy sequence and y_n is a subsequence of x_n, then y_n is a Cauchy sequence.

Theorem $(\mathbb{R}, d_2)$ is a complete a metric space.

Proof: Two facts about real analysis:

(i) Every real sequence has a (weakly) monotone subsequence,

(ii) If a real sequence is bounded and monotone, then it converges.

skipped.

$\square$

Theorem Let (X, d_X) and (Y, d_Y) be metric spaces. If (Y, d_Y) is complete then

$$(B(X,Y), d_\infty)$$

$\{f: X \to Y, \ f \text{ is bounded}\}$

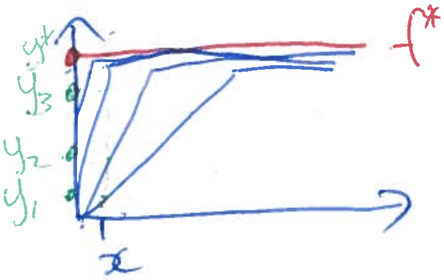
and $(CB(X,Y), d_\infty)$

$\{f \in B(X,Y) : f \text{ is continuous}\}$

are also complete.

$$d_\infty(f,g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Proof $B(X,Y)$ is complete: Let f_n be a Cauchy sequence in $(B(X,Y), d_\infty)$. Then for any $x \in X$, the sequence $y_n = f_n(x)$ is a Cauchy sequence in (Y, d_Y).



Since (Y, d_Y) is complete, y_n is a convergent sequence. Let's call the limit $f^*(x)$.

Since d_Y is continuous, we know that $d_Y(f^*(x), f_n(x)) = \lim_{m \to \infty} d_Y(f_m(x), f_n(x))$

for all $x \in X$ and all n.

Therefore,

$$d_Y(f^*(x), f_n(x)) \le \lim_{m\to\infty} \sup_{\hat{x}\in X} d_Y(f_m(\hat{x}), f_n(\hat{x}))$$

$$= \lim_{m\to\infty} d_\infty(f_m, f_n),$$

for all $x \in X$. So,

$$\sup_{\hat{x}\in X} d_Y(f^*(\hat{x}), f_n(\hat{x})) \le \lim_{m\to\infty} d_\infty(f_m, f_n)$$

$$\Rightarrow \quad d_\infty(f^*, f_n) \le \lim_{m\to\infty} d_\infty(f_m, f_n).$$

Since f_n is a Cauchy sequence, the RHS converges to zero as $n \to \infty$. So LHS does too, $d_\infty(f^*, f_n) \to 0$.

Remains to show that $f^* \in B(X, Y)$ and $f^* \in CB(X, Y)$, i.e. f^* is bounded and continuous. $\square$

2.3 Envelope theorem, value functions

$$\pi(p;w) = \max_{x \in \mathbb{R}^N_+} pf(x) - w \cdot x.$$

If $p\uparrow$ what happens to π?

$$\frac{\partial \pi(p;w)}{\partial p} ?$$

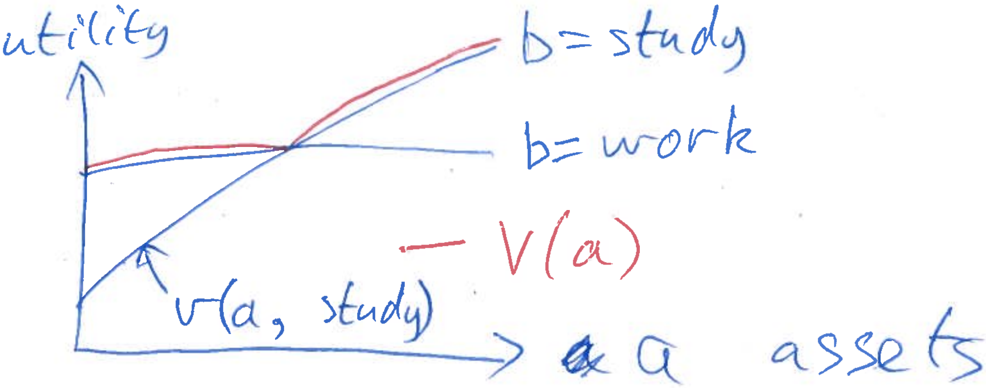
More abstract:

$$\underbrace{\underbrace{V(a)}_{\text{state}}}_{\text{value function}} = \max_{\underbrace{b}_{\text{choice}}} \underbrace{v(a, b)}_{\text{objective}} = v(a, \underbrace{b(a)}_{\text{policy}})$$

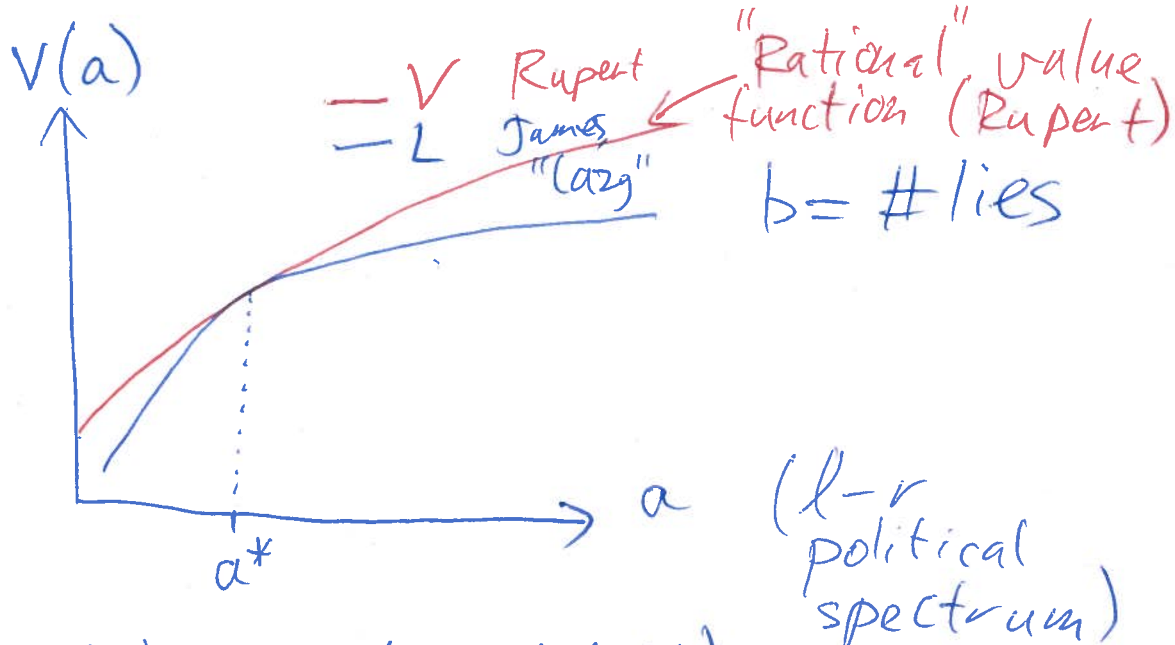
What is $\frac{\partial V}{\partial a}$?

Envelope Theorem Let $v : \underset{a}{\mathbb{R}^n} \times \underset{b}{\mathbb{R}^m} \to \mathbb{R}$ be a differentiable function, and let $V(a) = \max_b v(a,b)$ be its <u>upper envelope</u>, and let $b(a)$ be the <u>policy function</u>. If V is differentiable then

$$V'(a) = \left[\frac{\partial v(a,b)}{\partial a}\right]_{b=b(a)}.$$



Proof



Let $L(a) = v(a, \underbrace{b(a^*)}_{\text{last words}})$.

Note: $L(a) \leq V(a)$ for all a.

But $L(a^*) = V(a^*)$. Therefore, a^* minimises $V(a) - L(a)$, so we have the first-order condition

$$V'(a^*) - L'(a^*) = 0$$

$$\iff V'(a^*) = L'(a^*) = \left[\frac{\partial v(a,b)}{\partial a}\right]_{b=b(a^*)}$$

$\square$

Compare chain rule: $V'(a) = \left[\frac{\partial v(a,b)}{\partial a} + \underbrace{\cancel{\frac{\partial v(a,b)}{\partial b}}}_{=0} b'(a)\right]_{b=b(a)}$

Applying the envelope theorem to the profit function gives

$$\frac{\partial \pi(p;w)}{\partial p} = \left[\frac{\partial}{\partial p}\left[p f(x) - w\cdot x\right]\right]_{x=x(p;w)}$$

$$= \left[f(x)\right]_{x=x(p;w)}$$

$$= f(x(p;w))$$

$$= y(p;w) \leftarrow \text{output policy (supply function)}$$

$$\frac{\partial \pi(p;w)}{\partial w_i} = \left[\frac{\partial}{\partial w_i}\left[p f(x) - w\cdot x\right]\right]_{x=x(p;w)}$$

$$= \left[-x_i\right]_{x=x(p;w)}$$

$$= -x_i(p;w) \leftarrow \text{factor demand function}$$

Connect marginal values to optimal choices.

__Theorem 2.2__ Suppose $V(a) = \max_b v(a,b)$, where $v(\cdot, b)$ is a convex function for each b. Then V is a convex function.

varying a, holding b fixed