C6 Continuity

Def Consider two metric spaces (X, d_X) and (Y, d_Y). We say that $f: X \to Y$ is <u>continuous at $x^* \in X$</u> if for every sequence $x_n \in X$ with $x_n \to x^*$, the corresponding sequence $y_n = f(x_n)$ converges to $y^* = f(x^*)$. We say f is <u>continuous</u> if f is continuous at all points $x \in X$.

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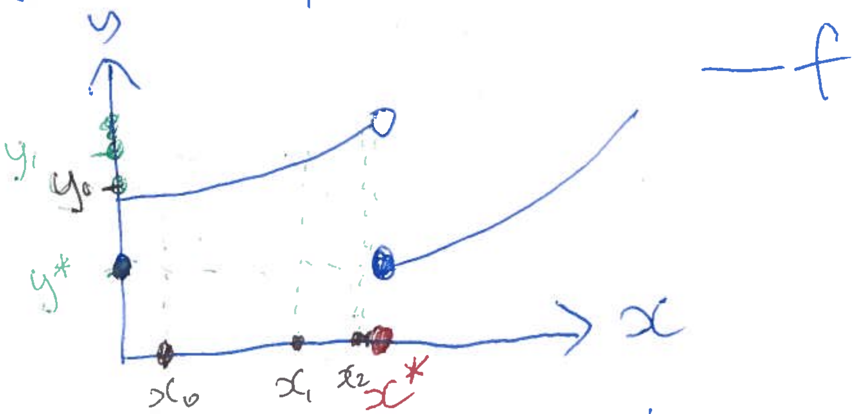

f is discontinuous because $x_n \to x^*$ but $f(x_n) \not\to f(x^*)$.

Assumed: $(X, d_X) = (\mathbb{R}_+, d_2)$ and $(Y, d_Y) = (\mathbb{R}, d_2)$.

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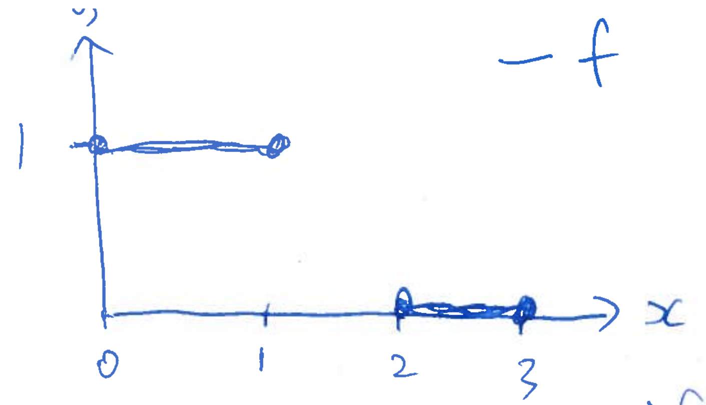
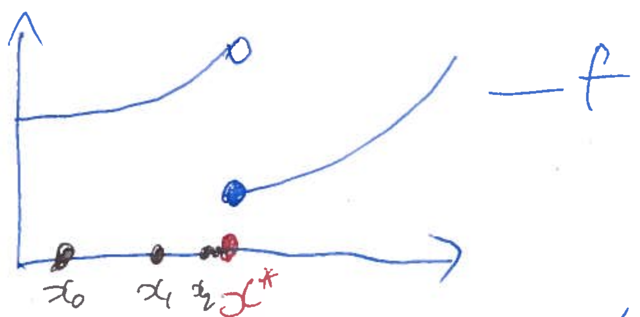


f is continuous, if $f: X \to Y$ where $X = [0,1] \cup [2,3]$ and $d_X = d_2$ and $(Y, d_Y) = (\mathbb{R}, d_2)$.

* If the domain's metric is the discrete metric, then the function is continuous.

eg: Redo the first example with ~~(X, d_X)~~ $X = \mathbb{R}_+$ and

$$d_X(x, x') = \begin{cases} 0 & \text{if } x = x' \\ 1 & \text{if } x \neq x'. \end{cases}$$



In this case $x_n \not\to x^*$, since $d(x_n, x^*) = 1$ for all n. f is continuous.

We now relate this definition to open and closed sets.

Notation:

* <u>image of a set:</u> $f(A) = \{f(a) : a \in A\}$.

* <u>pre-image of a set:</u> $f^{-1}(B) = \{x \in X : f(x) \in B\}$

or "inverse image"

note: $A \subseteq X$ and $B \subseteq Y$.

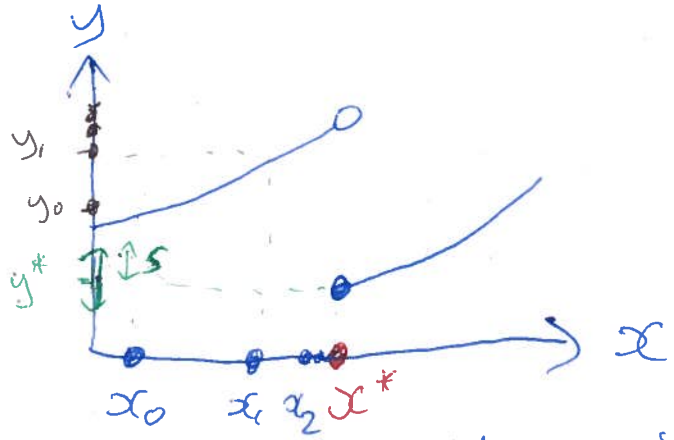
<u>Theorem C7</u> Let $f : X \to Y$ be a function between two metric spaces (X, d_X) and (Y, d_Y). Pick any $x^* \in X$ and let $y^* = f(x^*)$. Then f is continuous at x^* if and only if for every open ball $N_s(y^*) \subseteq Y$, there exists some open ball $N_r(x^*)$ such that

$$f(N_r(x^*)) \subseteq N_s(y^*).$$

<u>Proof:</u> $A =$ f is continuous at x^*
$B =$ there exists $N_r(x^*)$ ...

$B \Rightarrow A$. Same as "not A" $\Rightarrow$ "not B".
(contrapositive)

Suppose that for some $x_n \to x^*$, we have $y_n = f(x_n) \not\to y^*$.
We will find an open ball $N_s(y^*)$ such that every open ball ~~does~~ $N_r(x^*)$ ~~has~~ has $f(N_r(x^*)) \not\subseteq N_s(y^*)$.

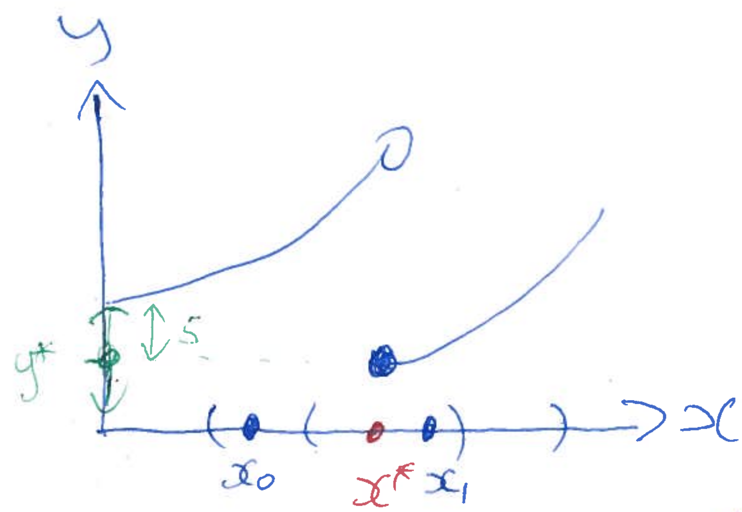


Since $y_n \not\to y^*$, there is some radius $s>0$ ~~st~~ s.t. no tail of y_n ~~l~~ lies inside $N_s(y^*)$. Since every open ball $N_r(x^*)$ contains a tail of x_n, it follows $f(N_r(x^*))$ contains a tail of y_n. Therefore $f(N_r(x^*)) \not\subseteq N_s(y^*)$ for all $r>0$.

"not B" $\Rightarrow$ "not A":

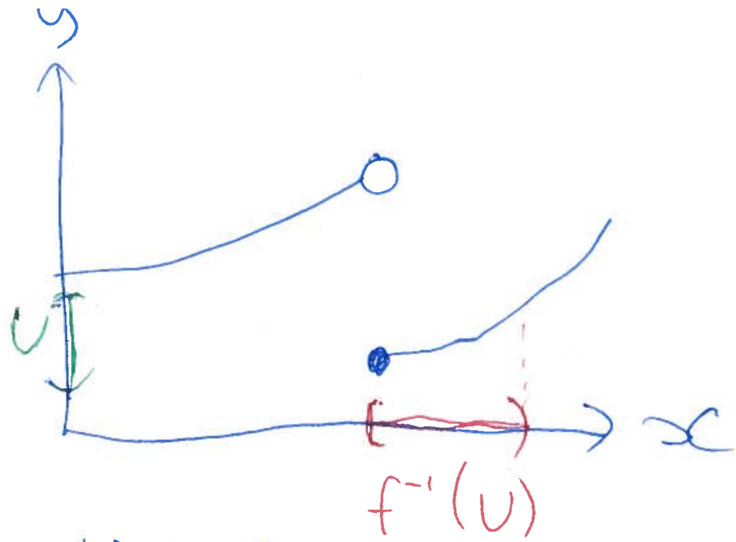
Conversely, suppose that for some open ball $N_s(y^*)$, there is no open ball $N_r(x^*)$ such that $f(N_r(x^*)) \subseteq N_s(y^*)$.

We will construct a sequence $x_n \to x^*$ such that $f(x_n) \not\to y^*$.
For every n, there exists some $x_n \in N_{\frac{1}{n}}(x^*)$ such that $f(x_n) \notin N_\delta(y^*)$.

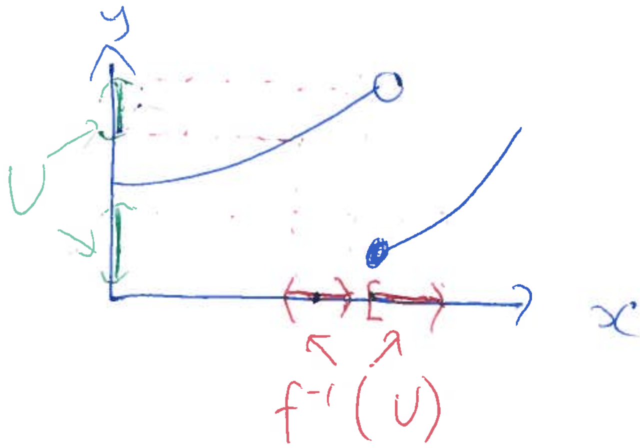


By construction, $d(x_n, x^*) < \frac{1}{n}$ so $x_n \to x^*$. But $f(x_n) \notin N_\delta(y^*)$ so $f(x_n) \not\to y^*$. □

Theorem 18 Let $f: X \to Y$ be a function between two metric spaces (X, d_X) and (Y, d_Y). Then f is continuous if and only if $f^{-1}(U)$ is an open set for all open sets $U \subseteq Y$.

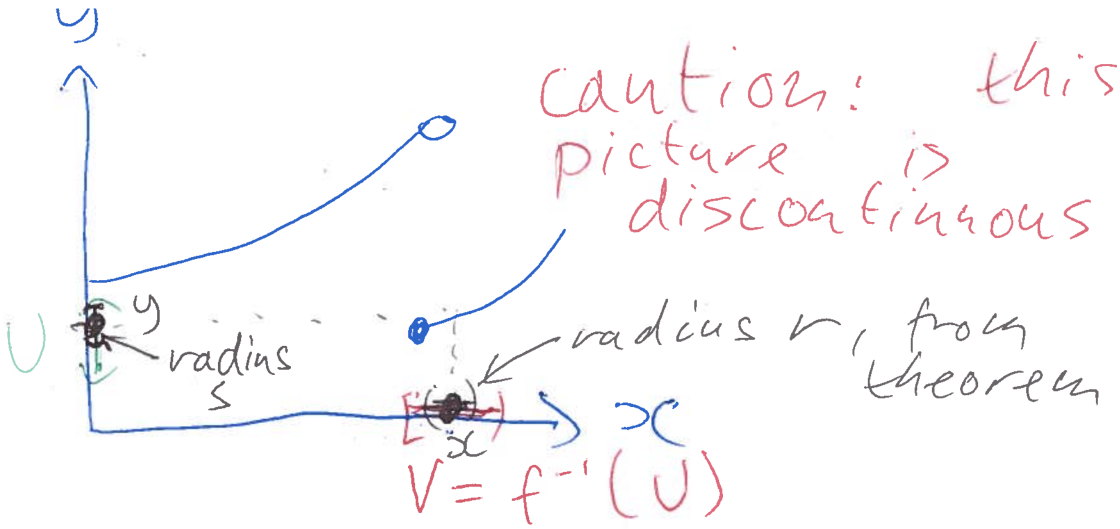


f is discontinuous because $f^{-1}(U)$ is not an open set.



<u>Proof</u> <u>Suppose f is continuous.</u>

Let U be any open set in (Y, d_Y), let $V = f^{-1}(U)$. Our goal: show that V is an open set in (X, d_X). This amounts to showing point $x \in V$ is an interior point. Pick any $x \in V$, and let $y = f(x)$. Since U is an open set and $y \in U$, then there is an open ball $N_\delta(y)$ such that $N_\delta(y) \subseteq U$.



By the previous theorem, since f is continuous, there exists some $N_r(x)$ such that $f(N_r(x)) \subseteq N_s(y)$.
It follows that $N_r(x) \subseteq V$.
So x is an interior point of V.
We conclude V is an open set.

skip second half. $\square$

2.2 Profit Maximisation

Notation:

$p \in \mathbb{R}_+$ output price

$w \in \mathbb{R}_+^{N-1}$ input prices (e.g. wages)

Profit function:

$$\pi(p;w) = \max_{\underbrace{x \in \mathbb{R}_+^{N-1}}_{\text{input quantities}}} \underbrace{pf(x)}_{\text{revenue}} - \underbrace{w \cdot x}_{\text{costs}}$$

shorthand: $\sum_{n=1}^{N-1} w_n x_n$

$$= pf(x(p;w)) - w \cdot \underbrace{x(p;w)}_{\text{policy function, factor demand function}}$$

Potential problems:

* There might be no optimal choice ($x(p;w)$ might not exist).
* There might be several optimal choices ($x(p;w)$ might not be unique). Rule this out if f is strictly concave.

Example 2.2

Notation:

w waste material input quantity

$g(w)$ glycerine output

$d(w)$ diesel output

p^w, p^g, p^d prices

$$\pi(p^g, p^d; p^w) = \max_{w} \; p^g g(w) + p^d d(w) - p^w w.$$