C9 Compact sets (cont'd)

Examples:

* $[0,1]$ in $(\mathbb{R}, d_2)$ is compact.
  closed & bounded in Euclidean space
* $[0,1]^2$ in $(\mathbb{R}^2, d_2)$ is compact.
* $[0,1] \cup [2,3]$ in $(\mathbb{R}, d_2)$.

Non-examples:

* $(0,1)$ in $(\mathbb{R}, d_2)$
* $((0,1), d_2)$
* $[0,\infty)$ in $(\mathbb{R}, d_2)$
* $(\mathbb{R}, d_2)$

not compact sets (inside $(\mathbb{R}, d_2)$)

not compact metric spaces

<u>Theorem</u> Suppose $f: X \to Y$ is a surjective and continuous function between (X, d_X) and (Y, d_Y). If (X, d_X) is compact, then (Y, d_Y) is compact.

Proof Let $y_n \in Y$. We want to prove y_n has a convergent subsequence.

Since f is surjective, there exists a sequence $x_n \in X$ such that $y_n = f(x_n)$. Since (X, d_X) is compact, x_n has a convergent subsequence x_{n_k}.

n_k says which items to select from x_n.
The k^{th} item in the subsequence is x_{n_k}.

Since f is continuous, $f(x_{n_k})$ is convergent. But $y_{n_k} = f(x_{n_k})$ is a subsequence of y_n. So y_n has a convergent subsequence. $\square$

~~The~~ Extreme Value Theorem

Suppose $f: X \to \mathbb{R}$ is a continuous function between (X, d_X) and $(\mathbb{R}, d_2)$. If (X, d_X) is compact and non-empty, then f has a maximum (and a minimum).

That is, the problem

$$\max_{x \in X} f(x)$$

has a solution.

<u>Proof</u> Set $Y = f(X)$. By the previous theorem, (Y, d_2) is a compact metric space, and hence Y is a compact set in $(\mathbb{R}, d_2)$. By the Bolzano-Weierstrass theorem, Y is closed and bounded. Since Y is ~~a~~ bounded, $\sup Y$ is finite. Let $y_n \in Y$ be

converging to sup Y. Since Y is closed, the limit, sup Y lies inside Y. So max Y exists. $\square$

eg: $X = [1, \infty)$ and $f(x) = -\frac{1}{x}$.

$$\sup_{x \in X} f(x) = 0.$$

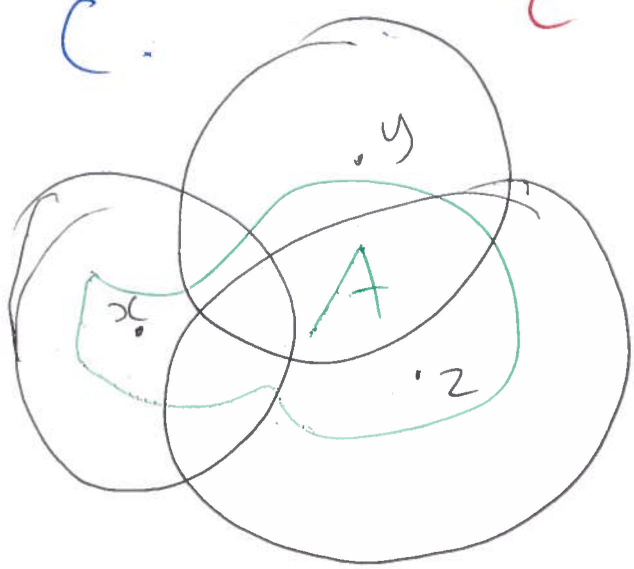
But there is no x^* s.t. $f(x^*) = 0$.

Range$(f) = [-1, 0)$ — not compact.

Look at compact sets in terms of open sets.

<u>Def</u> A <u>cover</u> of a set A is a collection of sets $\mathcal{C}$ such that
(set) ← curly C

$$A \subseteq \bigcup_{C \in \mathcal{C}} C.$$



$\mathcal{C} = \{ N_1(x), N_2(y), N_3(z) \}$.

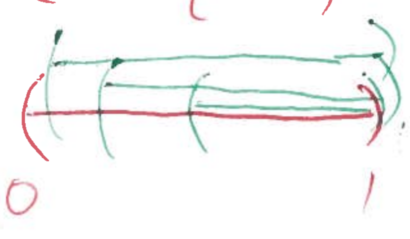
Def Let $\mathcal{C}$ be a cover of A. Then $\mathcal{C}'$ is a subcover of A if $\mathcal{C}' \subseteq \mathcal{C}$ and $\mathcal{C}'$ is a cover of A.

Def Let (X,d) be a metric space. An open cover of a set A is a cover $\mathcal{C}$ of A such that every $C \in \mathcal{C}$ is an open set.

Theorem Let (X,d) be a metric space. $A \subseteq X$ is compact if and only if every open cover of A has a finite subcover.

Eg: $(\mathbb{R}, d)$ and $A = (0,1)$.

Let $\mathcal{C} = \{(\frac{1}{n}, 1) : n \in \{2, 3, \ldots\}\}$.



There is no finite sub-cover.

Theorem (Heine-Borel)

Consider a Euclidean space $(\mathbb{R}^n, d_2)$ and a subset $X \subseteq \mathbb{R}^n$. Then X is closed and bounded if and only if every open cover of X has a finite subcover.

Proof

closed & bounded

Bolzano–Weierstrass

$\Longleftrightarrow$ compact

previous theorem

$\Longleftrightarrow$ every open cover has a finite subcover $\square$

Theorem (Cantor intersection theorem)

Let (X, d) be a metric space, and let K_n be a sequence of subsets of X. If each K_n is non-empty, compact, and nested (i.e. $K_{n+1} \subseteq K_n$) then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

__Proof__ Pick any sequence $x_n \in X$ such that $x_n \in K_n$. Since $x_n \in K_1$, and K_1 is compact x_n has a convergent subsequence. Without loss of generality, assume $x_n \to x^*$. Since K_1 is closed and $x_n \in K_1$, it follows that $x^* \in K_1$. Similarly $x^* \in K_n$. Therefore $x^* \in \bigcap_{n=1}^{\infty} K_n$. $\square$

C.11 Application: Extreme Punishments

Government choose

$s \in \mathbb{R}_+$ sanctions

$p \in [0,1]$ probability of conviction

$c(p)$ policing cost, $c : [0,1] \to \mathbb{R}$ strictly increasing & continuous

h honest pay-off

$-s$ convicted pay-off

b booty pay-off

$$\min_{s \in \mathbb{R}_+,\, p \in [0,1]} c(p)$$

$$\text{s.t. } h \geq -ps + (1-p)b.$$

Q: What are the possible ways of deterring crime?

Consider $f : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ defined by $f(s,p) = -ps + (1-p)b$.

The set of institutions that deter

crime is

$$D = f^{-1}\left((-\infty, h]\right).$$

Since f is continuous and $(-\infty, h]$ is a closed set (in the co-domain), we conclude that D is a closed set. But D is not compact.

Consider the sequence $(s_n, p_n) = \left(n(b-h), \frac{1}{n}\right)$. Note: $f(s_n, p_n) < h$, so $(s_n, p_n) \in D$.

Two problems:

* $p_n \to 0$, but $(s,0) \notin D$ for all s.
* (no police, torture to death, fails to deter crime)
* $s_n \to \infty$ — unbounded.

Reformulate:

$$\min_{(s,p) \in D} c(p).$$

Since D is not compact, the Extreme Value Theorem does not apply.

In fact there is no solution! Social welfare of proposal (s_n, p_n) is $-c(p_n)$. Since $p_n \to 0$, welfare $\to -c(0)$

Since c is strictly increasing, $-c(0)$ is the best possible welfare.

But deterring crime involves $p>0$, and hence welfare $< -c(0)$.

So there is no optimal solution, i.e. there is no minimum that "reaches" the infimum.