2.8 (i) Let $V(P) = \max_Q \pi(P, Q) = \max_Q TR(P, Q) - TC(Q)$. Then the envelope theorem establishes that

$$\begin{aligned} V'(P) &= \left[\frac{\partial}{\partial P}(PQ - TC(Q))\right]_{Q=Q(P)} \\ &= [Q]_{Q=Q(P)} \\ &= Q(P). \end{aligned}$$

It is not possible to use the envelope theorem to calculate the marginal revenue of a price increase (but it is possible with the chain rule – it is $Q(P) + PQ'(P)$).

(ii) The marginal profit, $V'(P)$, can also be calculated with the chain rule:

$$V'(P) = \left[\frac{\partial \pi(P, Q)}{\partial P} + \frac{\partial \pi(P, Q)}{\partial Q} Q'(P)\right]_{Q=Q(P)}.$$

The first term on the right is the "direct effect" – i.e. the extra revenue from the products that were previously sold. The second term on the right is the "indirect effect" – i.e. the extra revenue from the extra products that are sold after the price increase. The second term is zero.

2.10 (i) Let k be the knowledge the firm is endowed with. It chooses how much labour l and silicon s to buy at prices w and r, and sells $f(k, l, s)$ solar cells at price p. The firm's profit function is don't forget

$$\pi(k, p, w, r) = \max_{l,s} pf(k, l, s) - wl - rs. \tag{H.1}$$

all state vars don't forget $= pf(k, l(k), s(k)) - wl(k) - rs(k)$

(ii) Applying the envelope theorem, we calculate that

$$\begin{aligned} \frac{\partial \pi(k, p, w, r)}{\partial k} &= \left[\frac{\partial}{\partial k}(pf(k, l, s) - wl - rs)\right]_{l=l(k,p,w,r), s=s(k,p,w,r)} && \text{(H.2)} \\ &= [pf_k(k, l, s)]_{l=l(k,p,w,r), s=s(k,p,w,r)} && \text{(H.3)} \\ &= pf_k(k, l(k, p, w, r), s(k, p, w, r)). && \text{(H.4)} \end{aligned}$$

C.1 No. For example, consider the two functions, $f(x) = 0$ and

$$g(x) = \begin{cases} 0 & \text{if } x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Now, $d(f, g) = 0$ but $f \neq g$. This violates the first property of metric spaces.

Therefore, $x_n + y_n \to x^* + y^*$, as required.

This last step requires a proof: if $x_n \to x^*$ and $y_n \to y^*$, then we must prove that $x_n + y_n \to x^* + y^*$. Fix any $r > 0$. By the two conditions, there must be N_x and N_y such that

- $d_2(x_n, x^*) < r/2$ for all $n > N_x$, and
- $d_2(y_n, y^*) < r/2$ for all $n > N_y$.

Let $N = \max\{N_x, N_y\}$. Then, $|x_n - x^*| + |y_n - y^*| < r/2 + r/2$ for all $n > N$. We conclude that $|(x_n + y_n) - (x^* + y^*)| < r$ for all $n > N$.

C.33 Note that $f : X \to \mathbb{R}_+$; we will measure distances in the co-domain with d_1 (although it turns out that $d_1 = d_2$ for $\mathbb{R}^1$, so this is a purely cosmetic assumption). Fix any x_0. Then for all $x \in X$, the triangle inequality implies that

$$d(x, x_0) \leq d(x, x^*) + d(x^*, x_0)$$
$$d(x^*, x_0) \leq d(x, x^*) + d(x, x_0).$$

Rearranging gives

$$d(x, x_0) - d(x^*, x_0) \leq d(x, x^*)$$
$$d(x^*, x_0) - d(x, x_0) \leq d(x, x^*).$$

Putting these together, we deduce that

$$|f(x) - f(x^*)| = d_1(f(x), f(x^*)) \leq d(x, x^*).$$

Suppose $x_n \in X$ converges to x^*. Then $d_1(f(x_n), f(x^*)) \leq d(x_n, x^*) \to 0$. So $f(x_n) \to f(x^*)$.

not complete
$f : \mathbb{R}_{++} \to (0,1)$
$f(x) = \frac{x}{1+x}$

$[0,1)$

C.34 Consider the function $f : \mathbb{R}_+ \to (0, 1)$ defined by $f(x) = \frac{x}{1+x}$, where the domain and co-domain use the Euclidean metric. Now, f is continuous, the domain is complete, but the co-domain is not complete.

in (A,d)

C.35 Suppose $a_n \in A$ is a Cauchy sequence. Then a_n is also a Cauchy sequence in (X, d). Since (X, d) is complete, there is some point $a^* \in X$ such that $a_n \to a^*$. Since A is a closed set in (X, d), it follows that $a^* \in A$. We conclude that $a_n \to a^*$ in (A, d).

C.36 Suppose $z_n = (x_n, y_n)$ is a Cauchy sequence in (Z, d_Z). Then for all $r > 0$, there exists some N such that

- $d_Z(x_n, y_n; x_m, y_m) < r$ for all $n, m > N$,

C.34 $X = \mathbb{R}_+$ is complete

$Y = [0,1)$ is not complete ← $y_n = 1 - \frac{1}{n}$ is Cauchy, but not convergent.

(using d_2 in both cases)

$f: X \to Y$ defined by

$$f(x) = \frac{x}{1+x}.$$

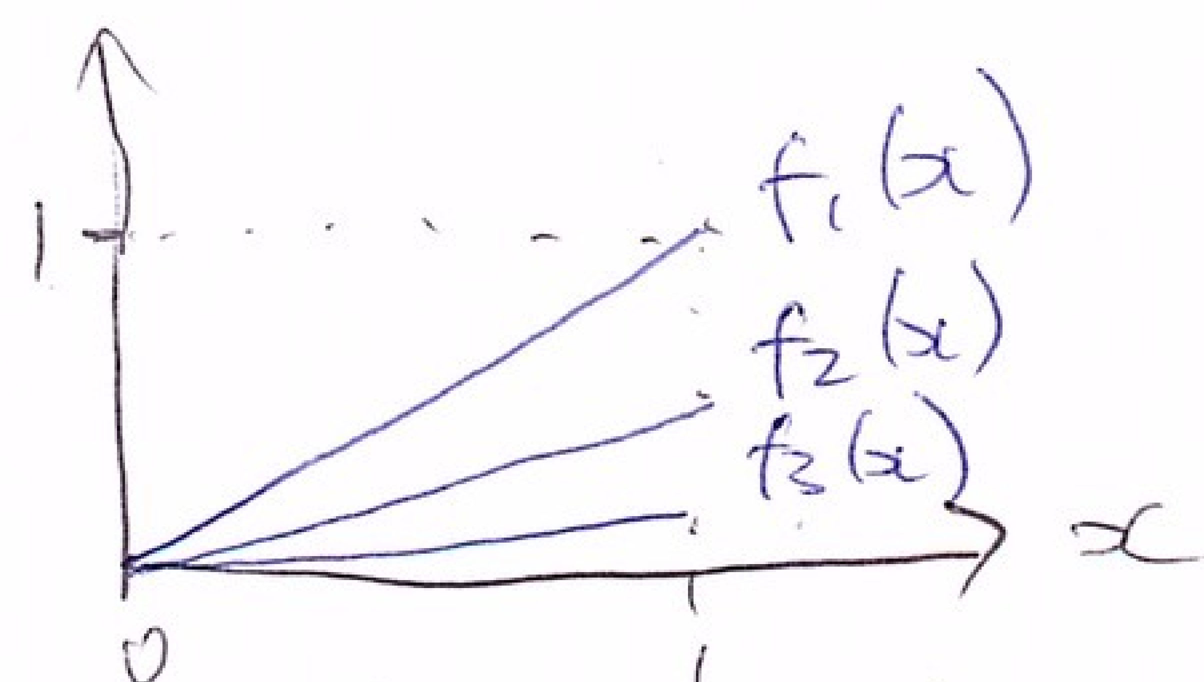
$f(X) = Y$: $f(0) = 0$ and $\lim_{x \to \infty} f(x) = 1$, so $1 \notin f(X)$.

f is continuous.

Common mistakes:

* $f: [0,1] \to (0,1)$, and $f(x) = x$, or $f(x) = \frac{1}{x}$: $f(0)$?

C.37



$f_1, f_2, f_3 \in X$

$f_n(x) = \frac{x}{n}$.

$f^*(x) = 0$.

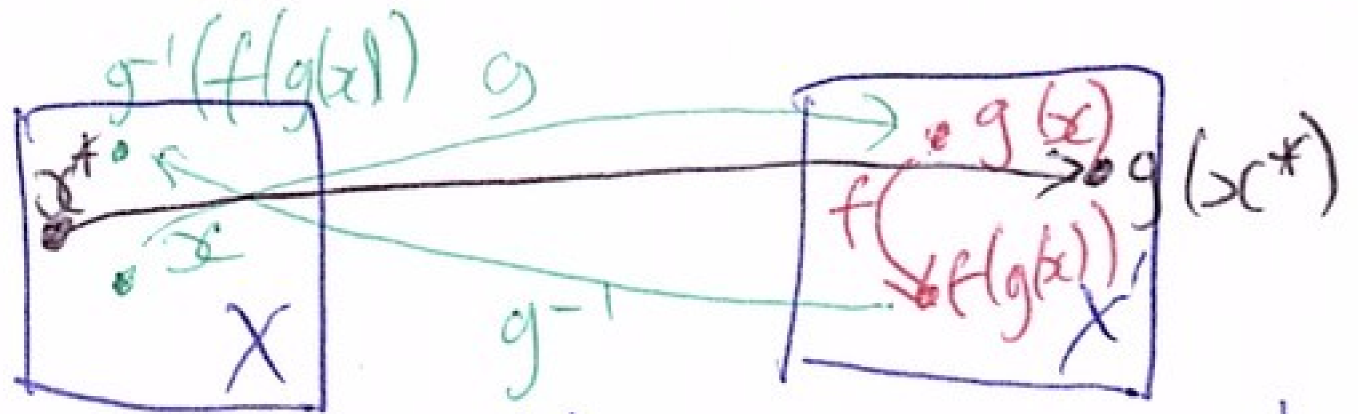
Inside $(B(X), d_\infty)$, $f_n \to f^*$ where $f^*(x) = 0$

But $f^* \in X$. So f_n is not convergent.

Def: $f: \mathbb{R} \to \mathbb{R}$ is <u>strictly increasing</u> if whenever $a < b$, $f(a) < f(b)$.

C.61 Consider $f: X' \to X'$ where f is continuous. We want to prove that f has a fixed point, i.e. $y^* \in X'$ such that $y^* = f(y^*)$.

Let $h: X \to X$ defined by $h(x) = \underbrace{g^{-1}(\underbrace{f(\overbrace{g(x)}^{\in X'})}_{\in X'})}_{X}$.



Now, h is continuous, so h has a fixed point $x^* \in X$. specifically,

$$x^* = h(x^*) = g^{-1}(f(g(x^*)))$$

$$\boxed{g(x^*)} = g(h(x^*)) = \cancel{g(g^{-1}}(f(g(x^*))))$$

$$= f(\boxed{g(x^*)})$$

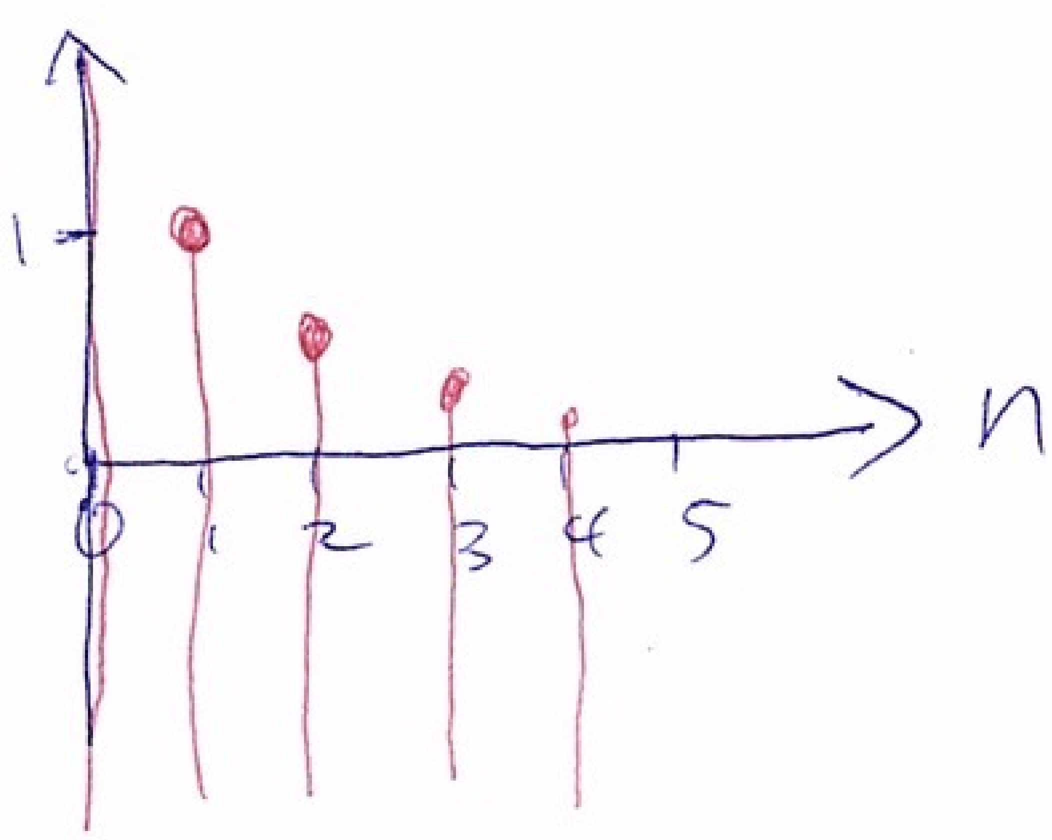
So $g(x^*)$ is a fixed point of f.

C.60 $X \subseteq B(\mathbb{N}, \mathbb{R}) = \{f: \mathbb{N} \to \mathbb{R},\ f \text{ is bounded}\}$

range(f) is contained in an open ball

eg: $x_n = 0$.. $(x_n) \in X$.

$x_n = \frac{1}{2}$. $(x_n) \notin X$ because $x_3 = \frac{1}{2} \leq \frac{1}{3}$



Yes, (X, d_∞) is complete. (Out of time)