$f(x_n) \not\to y^*$. For every n, there exists some $x_n \in N_{1/n}(x^*)$ such that $f(x_n) \notin N_s(y^*)$. Therefore, $x_n \to x^*$ but $f(x_n) \not\to y^*$. □

Theorem C.8. Let $f : X \to Y$ be a function between two metric spaces, (X, d_X) and (Y, d_Y). Then f is continuous if and only if $f^{-1}(U)$ is an open set for all open sets $U \subseteq Y$.

The theorem is illustrated in Figure C.9.

Proof. Suppose that f is continuous. Let U be any open set in (Y, d_Y), and let $V = f^{-1}(U)$. We need to show that V is an open set in (X, d_X). Consider any $x \in V$, and let $y = f(x)$. Since U is open and $y \in U$, there is some open ball $N_s(y) \subseteq U$. By Theorem C.7, there is some $N_r(x)$ such that $f(N_r(x)) \subseteq N_s(y) \subseteq U$. It follows that $N_r(x) \subseteq f^{-1}(f(N_r(x))) \subseteq f^{-1}(U)$. We conclude that $V = f^{-1}(U)$ is an open set.

Conversely, suppose that for all open sets $U \subseteq Y$, the set $f^{-1}(U)$ is open. We will show that f is continuous at every $x \in X$. Pick any x, let $y = f(x)$, and pick any open ball $U = N_s(y)$. Since U is an open set in (Y, d_Y), we know that $f^{-1}(U)$ is an open set. Therefore, there is some open ball $N_r(x) \subseteq f^{-1}(U)$ which implies $f(N_r(x)) \subseteq U = N_s(y)$. This means that Theorem C.7 applies, so we conclude that f is continuous at x. □

Question C.28. Let $f : X \to Y$ be a function between two metric spaces, (X, d_X) and (Y, d_Y). Prove that f is continuous if and only if $f^{-1}(A)$ is a closed set for all closed sets $A \subseteq Y$. *Hint: make use of the fact that A is closed if and only if $X \backslash A$ is open.*

Question C.29. Suppose that $u : \mathbb{R}^N_+ \to \mathbb{R}$ is a continuous utility function using Euclidean metrics for both the domain and co-domain. Prove that the indifference curves and upper contour sets of u are closed sets.

Question C.30. Prove that if $f : X \to Y$ is continuous and $g : Y \to Z$ is continuous, then $h : X \to Z$ defined by $h(x) = g(f(x))$ is continuous. (You should prove this for any metric for each of these three spaces.)

Question C.31. Let (X, d_X) and (Y, d_Y) be metric spaces, and consider any function $f : X \to Y$ such that there exists $y_0 \in Y$ such that for all $x \in X$, $f(x) = y_0$. Prove that f is continuous.

Question C.32. Prove that addition is continuous, i.e. that $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x + y$ is continuous, where the domain and co-domain use the Euclidean metric.

Question C.33. Let (X, d) be any metric space. Prove that for all $x_0 \in X$, the function $f(x) = d(x, x_0)$ is continuous.

Where $l(p, \phi, w)$ and $d(p, \phi, w)$ are the labour demand and wholesale food demand policies. Differentiating and multiplying by -1 on both sides gives

$$-\frac{\partial^2 \pi(p, \phi, w)}{\phi^2} = \frac{\partial d(p, \phi, w)}{\partial \phi}.$$

Since π is convex, the left side is negative. Thus, the right side is negative, so the sales policy is decreasing in the wholesale price ϕ.

The important lessons of this section are:

- The envelope theorem provides a formula for differentiating value functions, such as profit functions.
- The envelope formula provides a relationship between the derivative of the value function and the policy function. (Although we have not yet encountered the marginal cost curve coinciding with the supply curve.)
- If the decision-maker's problem is convex (i.e. satisfies all the convexity assumptions we need), then the value function is convex. This means the second derivatives (differentiating with respect to the same variable twice) of the value function are positive. This allowed us to deduce the signs of the derivatives of the policy function in the profit maximization problem.

Question 2.8. In classic undergraduate producer theory, profit π is a function of price P and output quantity Q,

$$\pi(P, Q) = TR(P, Q) - TC(Q),$$

where total revenue is $TR(P, Q) = PQ$, and $TC(Q)$ is the cost of producing Q.

(i) Use the envelope theorem to derive formulas for how revenue and profit change after a marginal price increase, i.e.

$$\frac{d}{dP}\pi(P, Q(P)),$$

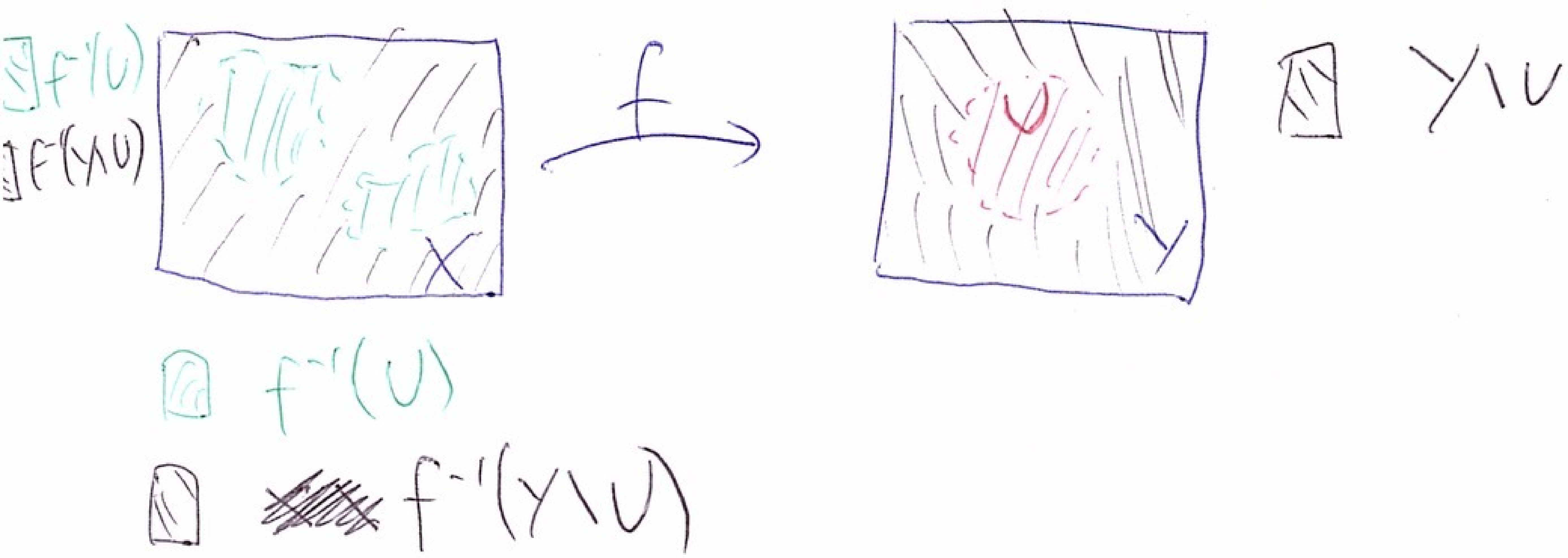
where $Q(P)$ is the output choice at price P. (Hint: if you are rusty on your calculus notation for total derivatives, you might find it helpful to write $g(P) = \pi(P, Q(P))$, and calculate the derivative $g'(P)$.)

(ii) Using algebra and words, explain the effect that the envelope theorem ruled out in part (i).

Question 2.9. Show that the firm's optimal policies are unresponsive to inflation. Show that inflation increases (nominal) profits. Do your answers suggest that a firm has an incentive to cause inflation (perhaps by bribing politicians)?

C.28 Why does $f^{-1}(U) = X \setminus f^{-1}(A)$?

$$= X \setminus f^{-1}(Y \setminus U)$$



C.29 Alternative. Let I be an ~~suppose~~ indifference curve. Specifically, $u(x)=u(y)$ for all $x,y \in I$.

Let $x_n \in I$ and suppose $x_n \to x^*$. We want to prove that $x^* \in I$.

Since u is continuous, $u(x_n) \to u(x^*)$.

Since $u(x_n) = u(x_1)$, that implies $u(x^*) = u(x_1)$ and hence $x^* \in I$.

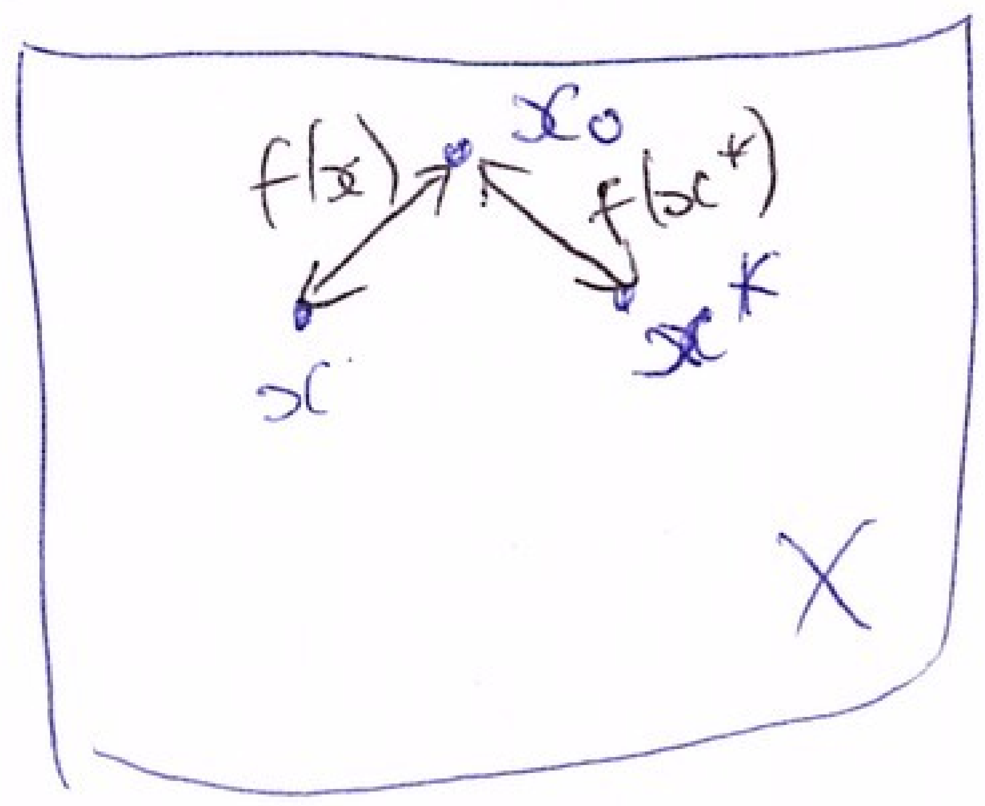
~~Wan~~ Fix $U \subseteq Z$ to be any open set.

We want to prove that $h^{-1}(U)$ is an open set.

~~Let~~ $h^{-1}(U) = f^{-1}(g^{-1}(U))$.

Let $V = g^{-1}(U)$. Since g is continuous, and U is open, V is open. Since f is continuous and V is open, $f\cancel{g}^{-1}(V)$ is open. So we conclude $h^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$ is open.

C.33



Sample solution to C.33 by Tracy Huiqin Shen.

(Edited for clarity by Andrew Clausen.)

Consider any sequence x_n in (X, d) such that $x_n \to x^*$. To establish that $f(x) = d(x, x_0)$ is continuous, we will prove that $f(x_n) \to f(x^*)$.

By the triangle inequality,

$$\begin{aligned} f(x_n) &= d(x_n, x_0) \leq d(x_n, x^*) + d(x^*, x_0) = d(x_n, x^*) + f(x^*), \text{ and} \\ f(x^*) &= d(x^*, x_0) \leq d(x^*, x_n) + d(x_n, x_0) = d(x_n, x^*) + f(x_n). \end{aligned}$$

This implies that

$$f(x_n) - f(x^*) \leq d(x_n, x^*) \text{ and } f(x^*) - f(x_n) \leq d(x_n, x^*)$$

and hence

$$|f(x_n) - f(x^*)| \leq d(x_n, x^*).$$

Because $x_n \to x^*$, for all $r > 0$, there exists N such that for all $n > N$, $d(x_n, x^*) < r$. This implies that for all $r > 0$, there exists N such that for all $n > N$, $|f(x_n) - f(x^*)| < r$. Therefore, $f(x_n) \to f(x^*)$.