Definition C.5. We say that y_n is a subsequence of x_n if there exists a sequence an increasing sequence $k_n \in \mathbb{N}$ (i.e. with $k_{n+1} > k_n$ for all n) such that $y_n = x_{k_n}$.

Theorem C.3. If $x_n \to x^*$ and y_n is a subsequence of x_n, then $y_n \to x^*$.

Proof. The condition $x_n \to x^*$ means that for every $r > 0$, there exists an $N \in \mathbb{N}$ such that $d(x_n, x^*) < r$ for all $n \geq N$. Since $y_n = x_{k_n}$ for some sequence k_n with $k_n \geq n$, it follows that $d(y_n, x^*) < r$ for all $n \geq N$. □

Question C.6. (Hard.) Prove that every sequence $x_n \in \mathbb{R}$ has a monotone (i.e. weakly increasing or decreasing) subsequence.

a sequence! in $(\mathbb{R}, d_2)$

Question C.7. Let (X, d) be any metric space, let x_n be a sequence in X, and let $x^* \in X$. Prove that if $\boxed{d(x_n, x^*)} \to 0$, then $x_n \to x^*$.

Answer. Let $y_n = d(x_n, x^*)$, which is a sequence of real numbers. The question is slightly ambiguous; it's not clear which metric space y_n lies in. It turns out the answer does not hinge on which metric is used; for simplicity we will use Euclidean space, i.e. $(\mathbb{R}, d_2)$.

Now, pick any $r > 0$. Then there exists some N such that:

- $y_n < r$ for all $n > N$, and hence
- $d(x_n, x^*) < r$ for all $n > N$.

We conclude that $x_n \to x^*$.

Question C.8. A household starts with no assets $a_0 = 0$, receives wages $w = 20$ every year, and has $\bar{c} = 10$ if non-discretionary consumption. Suppose that if the household has assets a_t in year t, they choose their next year's assets according to $a_{t+1} = \frac{4}{5}(w + a_t - \bar{c})$. Does the household's assets a_t converge to a steady state?

C.3 Boundaries

We now study the boundaries of sets inside metric spaces. Boundaries are important for several reasons:

- Many important ideas only make sense away from boundaries. For example, first-order conditions such as marginal benefit equals marginal cost are based on the idea of being able to both increase and decrease a choice a little bit, which is only possible away from a boundary.
- Optimal choices represent the boundary of possible utilities and profits. Therefore optimal choices often occur on the boundaries of the feasible options, such as the boundary of the set of affordable consumption bundles.

Intuitively, the boundary of a set is near points both inside and outside of the set. More precisely, the boundary of a set is defined as follows.

Definition C.6. Let A be any subset of a metric space (X, d). A point $x \in X$ is a **boundary point** of A if

(i) there exists a sequence $a_n \in A$ such that $a_n \to x$, and

(ii) there exists a sequence $b_n \in X \backslash A$ such that $b_n \to x$.

The set of boundary points of A is called the **boundary** of A, and is denoted ∂A.

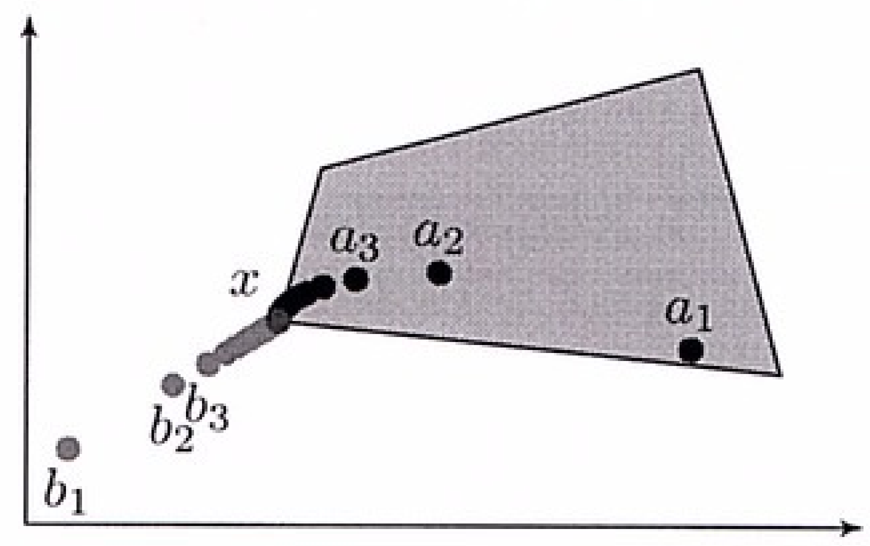


Figure C.6: A boundary point

Figure C.6 depicts a boundary point. Other examples of boundaries include:

- The boundary of $[0,1]$ in $(\mathbb{R}, d_2)$ is $\{0,1\}$.
- The boundary of $(0,1)$ in $(\mathbb{R}, d_2)$ is $\{0,1\}$.
- The boundary of $[0,1]$ in $([0,1], d_2)$ is $\emptyset$.
- The boundary of $(0,1)$ in $([0,1], d_2)$ is $\{0,1\}$.
- The boundary of $[0,1]$ in $(\mathbb{R}_+, d_2)$ is $\{1\}$.
- The boundary of $(0,1)$ in $(\mathbb{R}_+, d_2)$ is $\{0,1\}$.

To understand boundaries better, the following sections explore relationships to boundaries, including being inside a boundary and being away from a boundary.

Question C.9. Consider any price vector $p \in \mathbb{R}^N_{++}$. What is the boundary of the budget constraint, $A = \{x \in \mathbb{R}^N_+ : p \cdot x \leq m\}$ inside the metric space $(\mathbb{R}^N_+, d_2)$?

Answer. $\partial A = \{x \in \mathbb{R}^N_+ : p \cdot x = m\}$, assuming $p \in \mathbb{R}^N_{++}$.

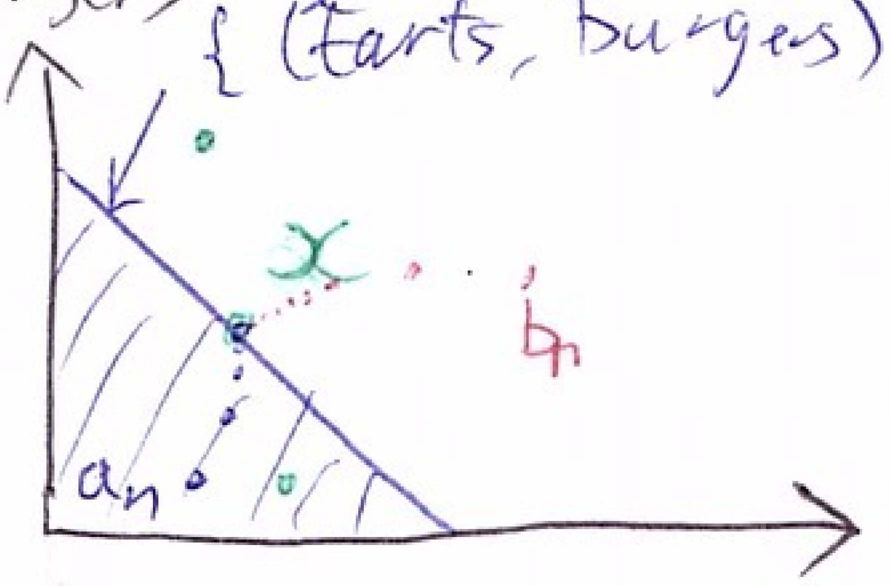
Proof. We check both requirements for $x \in \partial A$. The first requirement is that there exist a sequence $a_n \in A$ with $a_n \to x$. This is true if and only if $p \cdot x \leq m$.

The second requirement is that there exist a sequence $b_n \in \mathbb{R}^N_+ \backslash A$ with $b_n \to x$. This is true if and only if $p \cdot x \geq m$.

Both requirements are satisfied if and only if $p \cdot x \geq m$.

$p \cdot b_n > m$

burgers $\{(\text{tarts}, \text{burgers}) : p \cdot x = m\}$



A

tarts

$f: [0,1] \to \mathbb{R}_-$ f is bounded

(*) *Question* C.10. Let $A = \{f : [0, 1] \to \mathbb{R}, f(x) < 0 \text{ for all } x\}$. What is the boundary of A inside the metric space $(B[0, 1], d_\infty)$?

Answer. $\partial A = \{f : [0, 1] \to \mathbb{R}_-, f(x) = 0 \text{ for at least one } x \in [0, 1]\}$.

We check both requirements for $f \in \partial A$.

The first requirement is that there exist a sequence $a_n \in A$ such that $a_n \to f$. This is true if and only if $f(x) \leq 0$ for all $x \in [0, 1]$.

The second requirement is that there exist a sequence $b_n \in B[0, 1] \backslash A$ such that $b_n \to f$. This is true if and only if $f(x) \geq 0$ for at least one $x \in [0, 1]$.

Both requirements are met if $f(x) \leq 0$ for all $x \in [0, 1]$ and $f(x) = 0$ for some $x \in [0, 1]$.

Question C.11. Let (X, d) be any metric space where d is the discrete metric. Pick any set $A \subseteq X$. What is the boundary of A?

C.4 Closed Sets

Suppose a decision maker has a menu of M choices, and $x_n \to x^*$ is a convergent sequence of almost optimal choices, each better than the previous one. Is x^* on the menu? If not, then there might not be any optimal choice, which suggests that the decision-maker's problem has not been described accurately. To rule this problem out, we could assume that the menu M is closed, i.e. that it is impossible to escape from M by taking a limit. (This is analogous to – but of course completely different from – the idea of convexity, which is about escaping by drawing a line.) We will show that a set is closed if and only if it contains its boundary.

Definition C.7. Suppose A is a subset of a metric space (X, d). We say A is **closed** if there is no sequence $a_n \in A$ such that $a_n \to a^*$ and $a^* \notin A$.

For example,

- $[0, 1]$ is a closed set in $(\mathbb{R}, d_2)$.
- If (X, d) is any metric space, then X and $\emptyset$ are closed sets in (X, d).
- $(0, 1)$ is a closed set in $((0, 1), d_2)$, but *not* in $(\mathbb{R}, d_2)$.

Theorem C.4. Suppose A is a subset of a metric space (X, d). Then A is closed if and only if A contains its boundary, i.e. $\partial A \subseteq A$.

Proof. First, we show that if A is closed, then A contains its boundary. To see this, note that if $x \in \partial A$, then from the first part of the definition of boundary, there exists some sequence $a_n \in A$ such that $a_n \to x$. Since A is closed, we deduce that $x \in A$.

(f) Economic example of A: $\emptyset \neq \partial A \not\subseteq A$

Second, we show that if A contains its boundary, then A is closed. Specifically, we want to prove that if A contains its boundary, and $a_n \in A$ converges to x^*, then $x^* \in A$. Assume for the sake of contradiction that $x^* \notin A$. Then the sequence $b_n = x^*$ satisfies the properties that $b_n \notin A$ and $b_n \to x^*$. These two sequences a_n and b_n satisfy the definition that x^* is a boundary point of A. Since A contains its boundary, we conclude $x^* \in A$, violating the assumption. □

Definition C.8. Let A be in set inside a metric space (X, d). The **closure** of A is

$$\mathrm{cl}(A) = \{x^* \in X : \text{there is a sequence } x_n \in A \text{ with } x_n \to x^*\}.$$

Question C.12. Let (X, d) be any metric space. Prove that for any $A \subseteq X$, the set $\mathrm{cl}(A)$ is closed.

Question C.13. Let (X, d) be any metric space. Prove that for any set $A \subseteq X$, that $\mathrm{cl}(A) = A \cup \partial A$.

Answer. First, suppose $x \in \mathrm{cl}(A)$. Then there exists a sequence $a_n \in A$ such that $a_n \to x$. This implies one of two possibilities. One is that $x \in A$. The other is that $\boxed{x \in X \backslash A}$ so that the trivial sequence $b_n = x \in (X \backslash A)$ converges to x. The second possibility would imply that $x \in \partial A$. We conclude that x is either in A or ∂A.

$x \notin A$

Second, suppose that $x \in A \cup \partial A$. There are two possibilities, both of which imply that there is a sequence $a_n \in A$ with $a_n \to x$ and hence $x \in \mathrm{cl}(A)$. The first possibility is that $x \in A$. In this case, the trivial sequence $a_n = x \in A$ converges to x. The second possibility is that $x \in \partial A$. The definition of boundary points implies that there exists some sequence $a_n \in A$ such that $a_n \to x$.

Question C.14. Consider any price vector $p \in \mathbb{R}^N_{++}$. Is the budget constraint, $A = \left\{x \in \mathbb{R}^N_+ : p \cdot x \leq m\right\}$ a closed set inside the metric space $(\mathbb{R}^N_+, d_2)$?

Question C.15. Let (X, d) be any metric space. Prove that if $A \subseteq X$ is a finite set, then A is closed.

Answer. Suppose that $x \in \partial A$. Then there exists a sequence $a_n \in A$ such that $a_n \to x$. Since A is a finite set, there is a smallest possible distance between points in A; let's call that distance r. Since $a_n \to x$, there must be some N such that

- $d(a_n, x) < r$ for all $n > N$, and hence
- $a_n = x$ for all $n > N$.

Therefore, $x \in A$. We conclude that $\partial A \subseteq A$ and so A is a closed set.

Question C.16. Prove that if A and B are closed sets inside the metric space (X, d), then $A \cup B$ is a closed set.

Question C.17. Provide a counter-example to the following hypothesis: the union of a collection of closed sets is closed.

Answer. Consider the metric space $(\mathbb{R}, d_2)$ and the sets, $A_n = [0, 1 - 1/n]$. The union of all these sets is $[0, 1)$, which is not closed.

Question C.18. Prove that if $\mathcal{A}$ is a set of closed sets inside the metric space (X, d), then $\cap_{A \in \mathcal{A}} A$ is also a closed set.

Question C.19. Let (X, d) be a metric space. Prove that

$$\mathrm{cl}(A) = \cap \{C \subseteq X : C \text{ is closed, } A \subseteq C\}.$$

C.5 Open Sets

Sometimes it is important to focus attention on points that are away from the boundary of a set. For example, first-order conditions require thinking about choices that can be both increase or decreased.

A set is open if every point inside it is distant from the exterior (complement) of the set. We will later say that open sets do not contain any of their boundaries. The usual way to formalise this idea is in terms of open balls.

Definition C.9. The **open ball** centred at x with radius r in the metric space (X, d) is $N_r(x) = \{y \in X : d(x, y) < r\}$.

Definition C.10. Suppose A is a subset of a metric space (X, d). We say a point $x \in A$ is an **interior point** if there is an open ball $N_r(x)$ such that $N_r(x) \subseteq A$. The set of interior points of a set A is called the **interior** of A. We say A is an **open set** if it equals its interior. If A is an open set, and $x \in A$, then we say that A is an **open neighbourhood** of x.

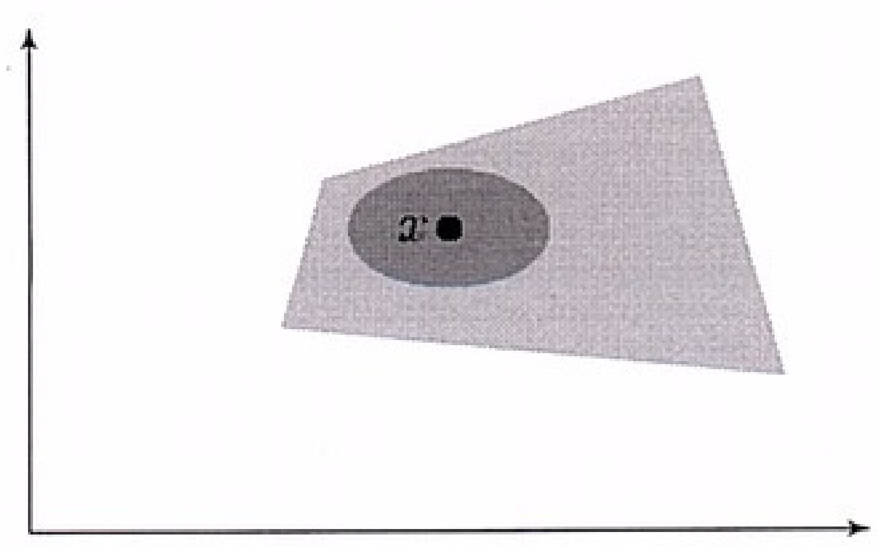
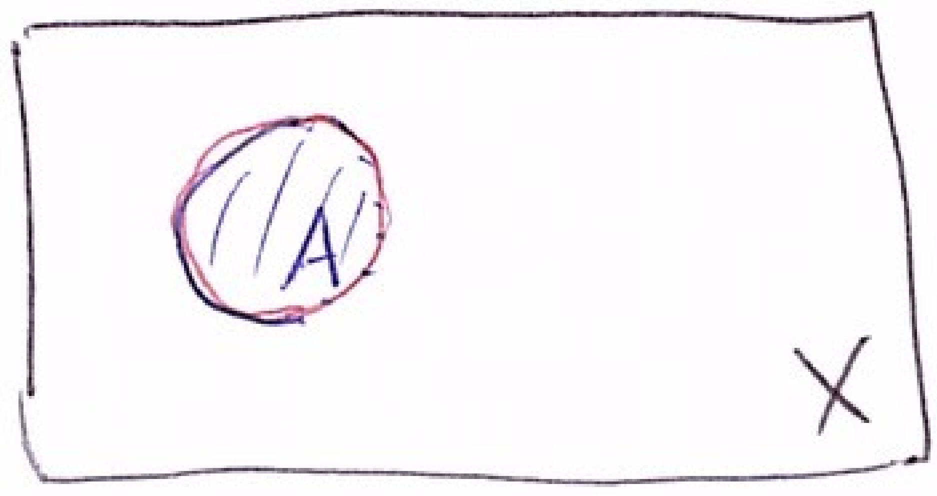


Figure C.7: A set is open if every element x is contained inside a ball inside the set

The definition of an open set is illustrated in Figure C.7. Examples and non-examples include:

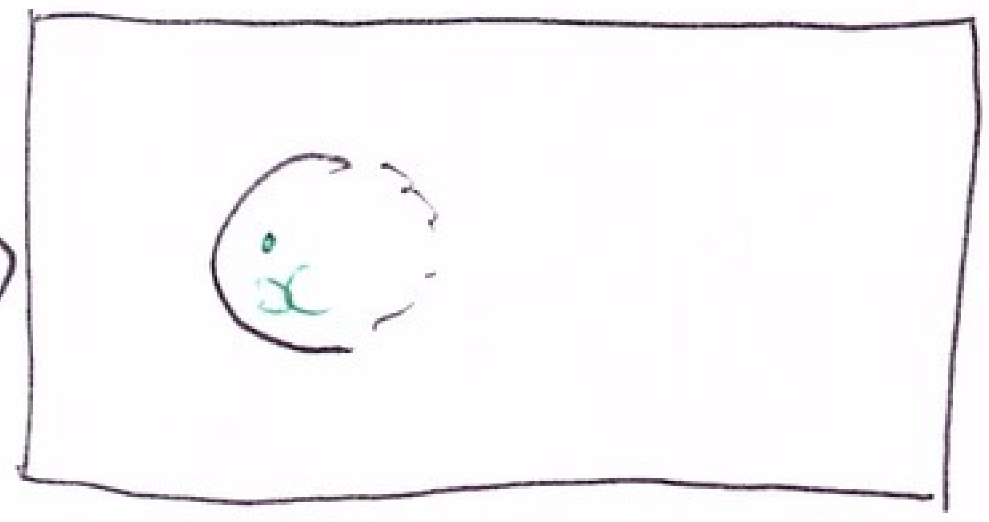
C.13



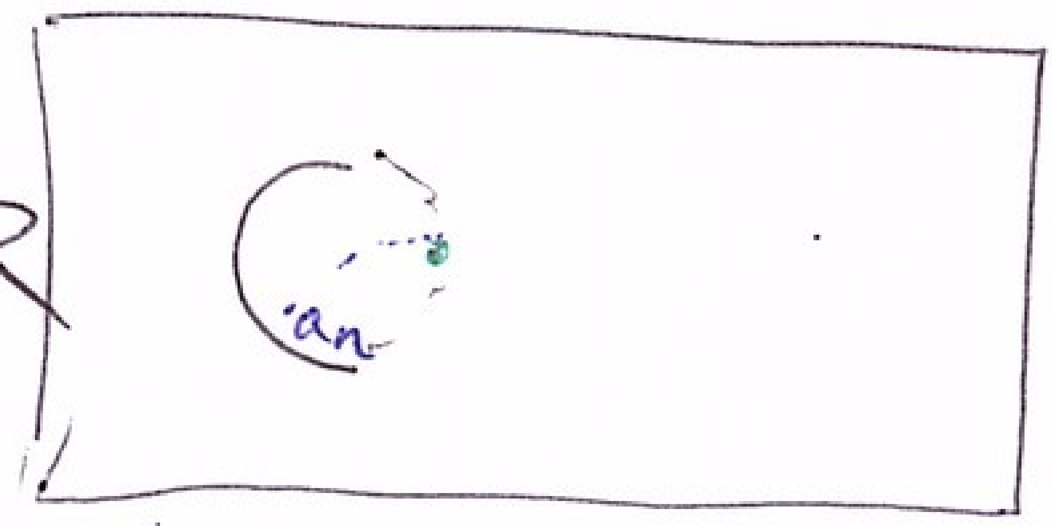
∂A —

$cl(A) = A \cup \partial A$

$x \in cl(A) \Rightarrow$



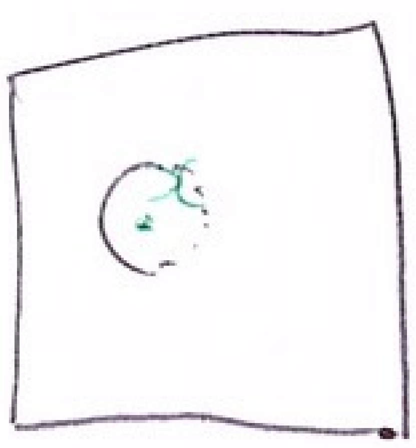
OR



$x \in A$

$x \notin A \Rightarrow b_n = x \notin A$
$\Rightarrow x \in \partial A$

$x \in A \cup \partial A$



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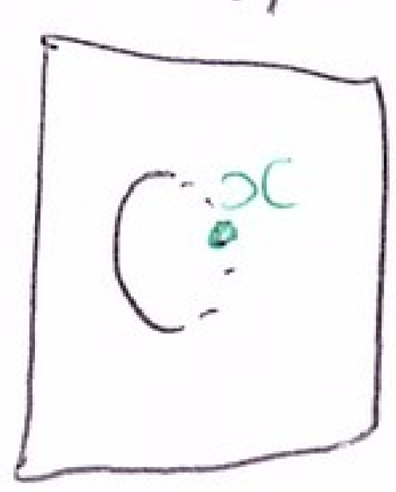
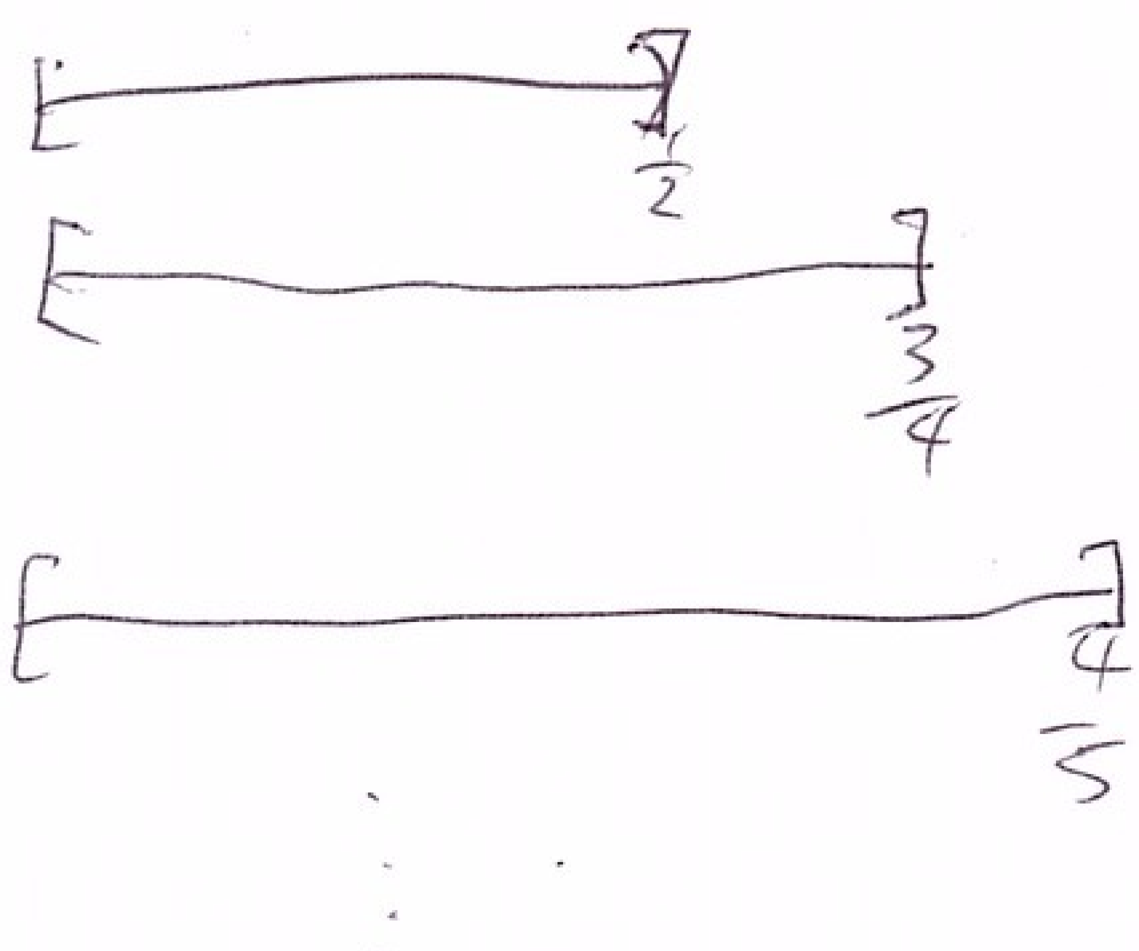


$\Rightarrow \quad x \in cl(A)$

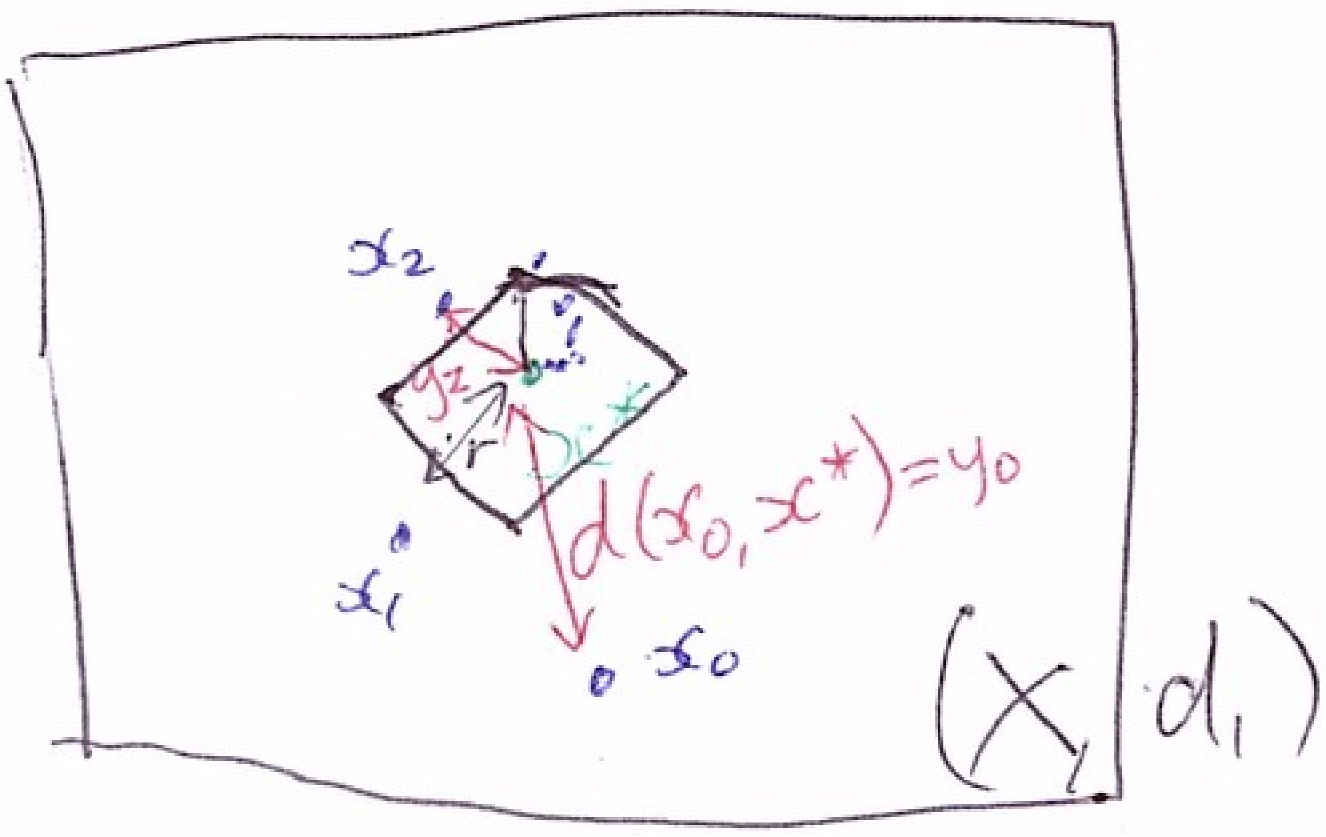
$b_n = x$

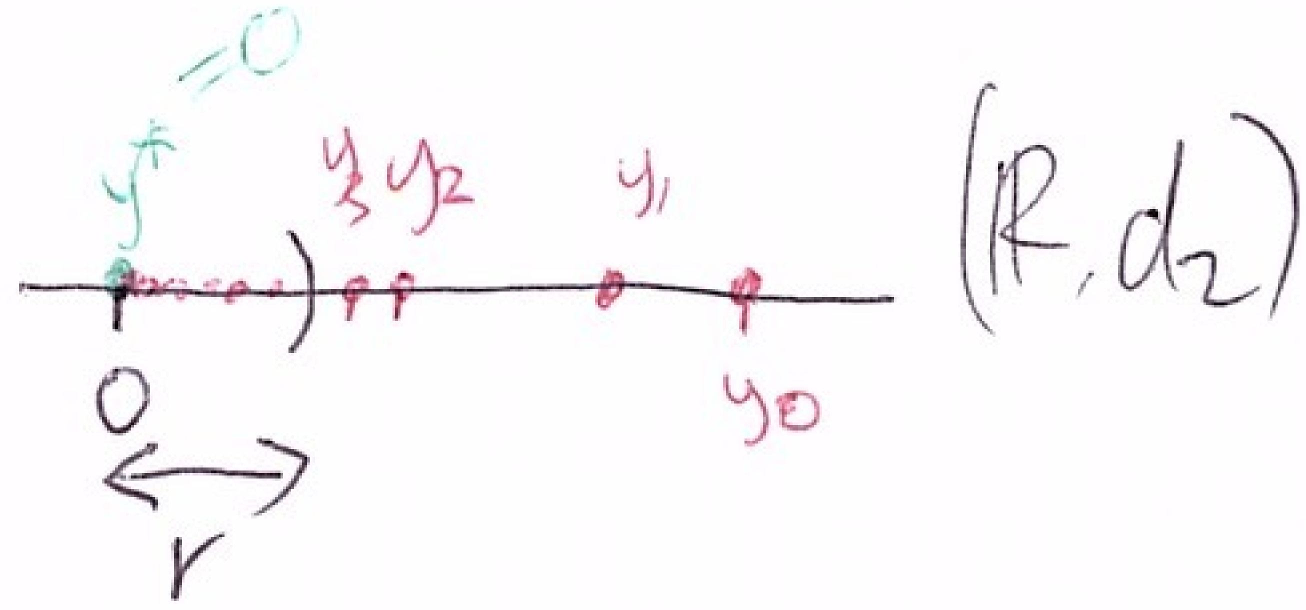
C.17



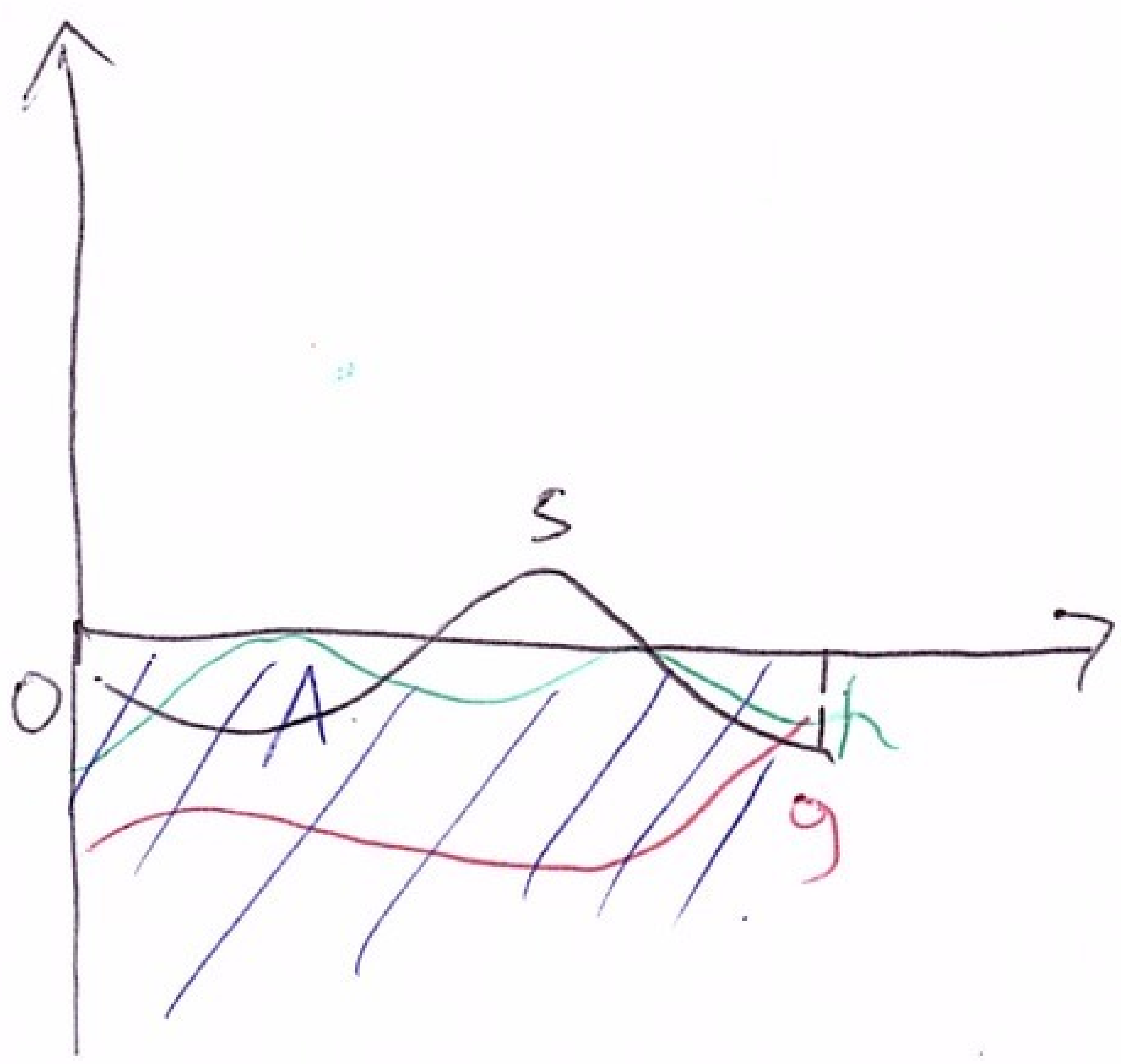
Union:

C.7





C.10



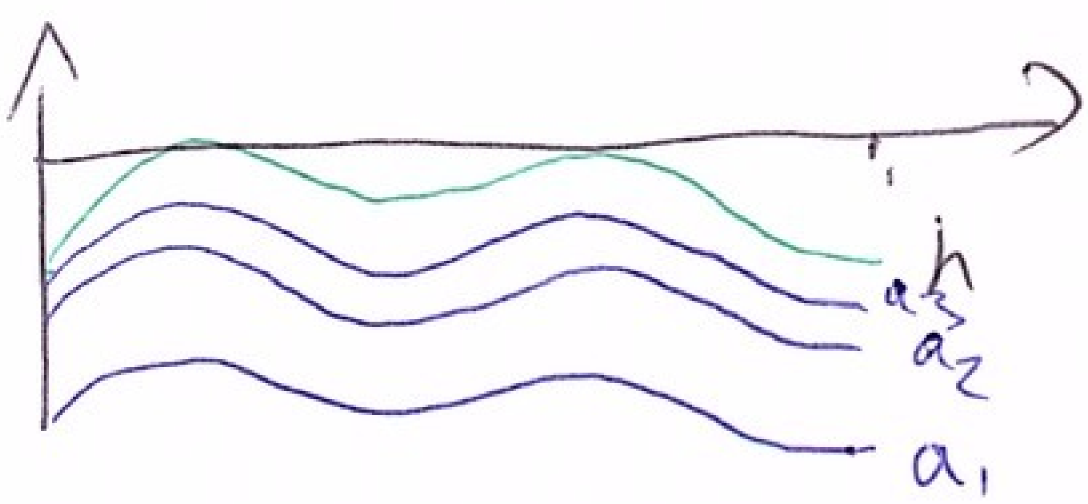
e.g. $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$

$f \notin B[0,1]$

$g \in A \subseteq B[0,1]$

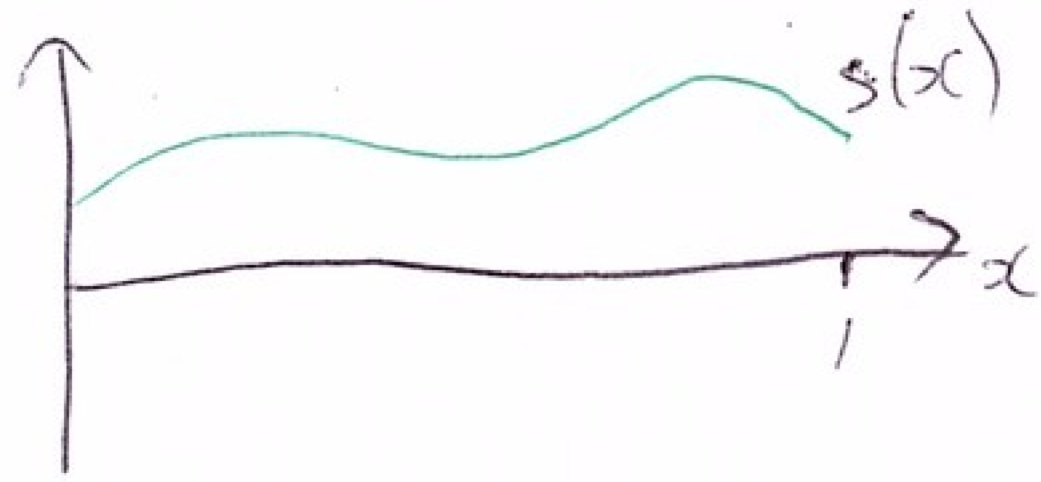
$h \in \partial A$, $h \notin A$

$s \notin A$



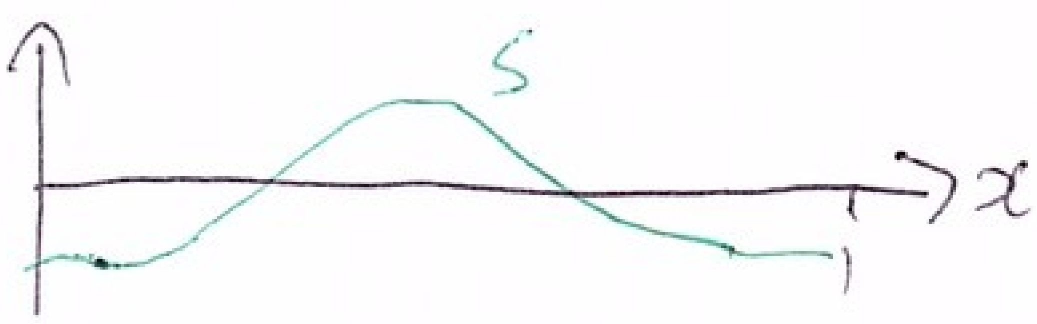
$a_n(x) = h(x) - \frac{1}{n}$

$d(a_n, h) = \frac{1}{n}$

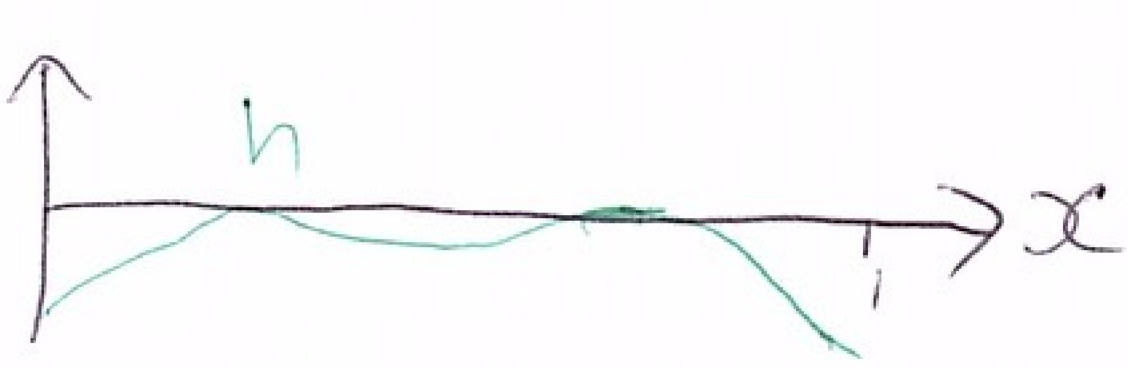


$b_n = s$ for all n

since $s \notin A$, $\Rightarrow b_n \notin A$.



$s \notin A$, so $b_n = s$



$h \notin A$, so $b_n = h$

No

Date

Tutorial 7 attempt.

C7. Since $d(x_n, x^*) \to 0$

why?

$\Rightarrow$ for every $r > 0$, there exist an $N \in \mathbb{N}$ such that $d(x_n, x^*) < r$ for all $n \geq N$.

The sequence is convergent towards the limit x^* and a sequence can converge to at most one point in X.

$\therefore x_n \to x^*$ $\square$.

C9. Since both x and p are ≥ 0, $A \geq 0$.

wrong space

M is a boundary point of A since

(i) there exist a sequence $a_n \in A$ such that $a_n \to M$

(ii) there exist a sequence $b_n \in \mathbb{R}^N_+ \setminus A \to M$.

0 is not a boundary point because (ii) is not satisfied.

$\therefore$ boundary of A is $\{M\}$

wrong space

C10 0 is a boundary point of A

(i) there exist a sequence $a_n \in A$ such that $a_n \to 0$

(ii) there exist a sequence $b_n \in B[0,1] \setminus A$ such that $b_n \to 0$

$\therefore$ boundary of A is $\{0\}$.

No.

Date :

C13 let $x \in \partial A$

since $x \in \partial A$, there exist some sequence $a_n \in A$ such that $a_n \to x$. We conclude $x \in cl(A)$ (since A is closed if no sequence $a_n \in A$ such that $a_n \to a^*$ and $a^* \notin A$). yes

$\therefore$ If A is closed, it contains ∂A $(\Rightarrow cl(A) = A \cup \partial A$. $\square$

C15. let $A = \{x_1, \ldots, x_n\}$. We can express A as $\bigcup_{i=1}^{n} \{x_i\}$. Each singleton is closed so the union of finitely many closed sets is still closed. $\square$

C17.

$A_n = [\frac{1}{n}, \infty)$.

Homework 3

08 October 2017 15:23

Question C.7. Let (X, d) be any metric space, let x_n be a sequence in X, and let $x^* \in X$. Prove that if $d(x_n, x^*) \to 0$, then $x_n \to x^*$.

Proof : Let : $r_n = d(x_n, x^*)$.

$\Rightarrow\ r_{n-1} > d(x_n, x^*) : r_{n-1} \in \mathbb{R}_{++}$ (as $d(x_n, x^*) \to 0$). ?

$\Rightarrow\ \exists N = \lceil r_{n-1} \rceil \in \mathbb{N} : r_{n-1} > d(x_n, x^*)\ \forall n \geq N$. □ ?

Question C.9. Consider any price vector $p \in \mathbb{R}^N_{++}$. What is the boundary of the budget constraint, $A = \{x \in \mathbb{R}^N_+ : p \cdot x \leq m\}$ inside the metric space $(\mathbb{R}^N_+, d_2)$?

Boundary of A, $\partial A : \{x \in \mathbb{R}^N_+ : p \cdot x = m\}$ but why?

$\{x \in \mathbb{R}^n_+ : p \cdot x = 0\} \notin \partial A$, as $\nexists\, x_n \to 0 : x_n \in \mathbb{R}^N_+ \setminus A$.

true

Question C.10. Let $A = \{f : [0,1] \to \mathbb{R}, f(x) < 0 \text{ for all } x\}$. What is the boundary of A inside the metric space $(B[0,1], d_\infty)$?

wrong set

$\partial A = \{0\}$

Question C.13. Let (X, d) be any metric space. Prove that for any set $A \subseteq X$, that $\mathrm{cl}(A) = A \cup \partial A$.

Proof : $cl(A) = A \cup \partial A \Rightarrow \left(x \in cl(A) \Leftrightarrow x \in A \vee x \in \partial A \right)$.

$\rightarrow$: Suppose, $x \in cl(A) \not\Rightarrow x \in A \vee x \in \partial A$.

$\Rightarrow \exists\, x_n \in A : x_n \to x$, where $x \notin A \vee \partial A$.

$\Rightarrow x : d(x_n, x) \not\to 0 \Rightarrow x_n \not\to x$. ↯ ?

$\leftarrow$: $\forall x \in A\ \exists\, x_n \in A : x_n = x\ \forall n \in \mathbb{R}$.

$\Rightarrow x \in A \Rightarrow x \in cl(A)$.

$\forall x \in \partial A\ \exists\, x_n \in A : x_n \to x$, by definition of ∂A.

$\Rightarrow x \in \partial A \Rightarrow x \in cl(A)$. □

Question C.15. Let (X, d) be any metric space. Prove that if $A \subseteq X$ is a finite set, then A is closed.

Proof : A is finite $\Rightarrow A = \{x_1, x_2, x_3 \cdots x_k\}$ where $k < \infty$.

Let $B_i := \{x_i\}\ \forall i \in [1, k]$.

$\Rightarrow \exists!\ x_n \in B_i$, i.e. $x_n = x_i\ \forall n \in \mathbb{N}$.

$\Rightarrow \partial B_i = \{x_i\} \Rightarrow \partial B_i \subseteq B_i \Rightarrow B_i$ is closed $\forall i \in [1, k]$.

Let $B_i = \{x_i\}$ $\forall i \in [1,k]$.

$\Rightarrow \exists!\ x_n \in B_i$, i.e. $x_n = x_i$ $\forall n \in \mathbb{N}$.

$\Rightarrow \partial B_i = \{x_i\} \Rightarrow \partial B_i \subseteq B_i \Rightarrow B_i$ is closed $\forall i \in [1,k]$.

Lemma: B_i is closed $\forall i \in k<\infty \Rightarrow \bigcup_{i=1}^{k<\infty} B_i$ is closed.

X

$\therefore A = \bigcup_{i=1}^{k<\infty} B_i \wedge B_i$ is closed $\forall i \in [1,k] \Rightarrow A$ is closed. □

Proof of Lemma: Without loss of generality,

let: $B_1 = \{x_1\} \wedge B_2 = \{x_2\} : x_1, x_2 \in X$.

$\partial B_1 = \{x_1\} \subseteq B_1 \wedge \partial B_2 = \{x_2\} \subseteq B_2$ in (X,d).

$\Rightarrow B_1 \cap B_2 = \{x_1, x_2\} \wedge \partial(B_1 \cap B) = \{x_1, x_2\} \subseteq B_1 \cap B_2$.

$\Rightarrow B_1, B_2$ are closed $\Rightarrow \bigcup_{i=1}^{2} B_i$ is closed.

Question C.17. Provide a counter-example to the following hypothesis: the union of a collection of closed sets is closed.

$(0,1) = \{x_1, x_2, \dots, x_n\}$ where $n = \infty$.

$(0,1) = \bigcup_{i=1}^{\infty} A_i$, where $A_i = \{x_i\}$ $\forall i \in n$.

∂A_i in $(\mathbb{R}, d_2)$: $\partial A_i = \{x_i\} \subseteq A_i \Rightarrow A_i$ is closed.

However $(0,1)$ is not closed in $(\mathbb{R}, d_2)$.

✓