$\boxed{C.2}$

(i) $d'(x,y)=0 \iff x=y$

a compact proof

<u>Proof:</u> $d'(x,y)=0$

$$\iff \frac{d(x,y)}{1+d(x,y)} = 0$$

$$\iff d(x,y) = 0\times(1+d(x,y))$$

$$\iff d(x,y) = 0$$

$$\iff x = y,$$ since (X,d) is a metric space. $\square$

2017年10月4日 14:20

C2 If $d'(x,y)=0$, then $d(x,y)=0$. Since (X,d) is a metric space, $x=y$

If $x=y$, then $d(x,y)=0$ since (X,d) is a metric space. If $d(x,y)=0$, then $d'(x,y)=0$

"If $d'(x,y)=0$ then $x=y$." missing?

It has been shown that $d'(x,y)=0$ iff $x=y$.
The first property is satisfied.

$$d'(x,y)=\frac{d(x,y)}{1+d(x,y)}=\frac{d(y,x)}{1+d(y,x)} \text{ (since } (X,d) \text{ is a metric space)}$$
$$=d'(y,x)$$

Hence the second property is satisfied.

Let $b=d(x,y)$ denote the distance between two points $x,y\in X$ and $c=d(x,z)$ denote the distance between two points $x,z\in X$

big jump!

$$d'(x,y)+d'(x,z)=\frac{d(x,y)}{1+d(x,y)}+\frac{d(x,z)}{1+d(x,z)}\geq\frac{d(x,y)+d(x,z)}{1+d(x,y)+d(x,z)} \text{ (from hint)}$$
$$\geq\frac{d(x,z)}{1+d(x,z)} \text{ (property of metric space)}$$
$$=d'(x,z)$$

Hence the third property is satisfied.

Since (X,d') satisfies the three property of metric space, (X,d') is a metric space by definition □

C3

If $x=y$, then $d(x,y)=0$ (since (X,d) is a metric space)

$d(x,x)=0$ for all $x\in X$

Since $\min\{1,0\}=0$, $d'(x,y)=d(x,y)=0$

If $d(x,y)=0$, then $d(x,y)=d'(x,y)$, then $x=y$. (since (X,d) is a metric space)

$\therefore d'(x,y)=\min\{1,d(x,y)\}=\min\{1,0\}=0$

If $d(x,y)\leq 1$,
$d'(x,y)=d(x,y)=d(y,x)=d'(y,x)$ (since (X,d) is a metric space)
If $d(x,y)>1$, then $d(y,x)>1$
$\Rightarrow d'(x,y)=1$ and $d'(y,x)=1$
Hence $d'(x,y)=d'(y,x)$ for all $x,y\in X$

$d'(x,y)+d'(x,z)=\min\{1,d(x,y)\}+\min\{1,d(x,z)\}$

highlight $d'(y,z)=\min\{1,d(y,z)\}$

If $d'(x,y)+d'(x,z)=d(x,y)+d(x,z)$,

then $d'(x,y)+d'(x,z)\geq d(\boxed{y,z})\geq d'(\boxed{y,z})$ ← d' "inherits" the triangle inequality

Four cases?

lost!

If $d'(x,y)+d'(x,z)=2$, then $d'(y,z)=\begin{cases} d(y,z) & \text{if } d(y,z)\leq 1, \\ 1 & \text{if } d(y,z)>1 \end{cases}$

which in both case $d'(y,z)\leq 1\leq 2$

If $d'(x,y)+d'(x,z)=1+d(x,y)\geq\min\{1,d(y,z)\}=d'(y,z)$ (since $d(x,y)\geq 0$)

The same applies when $d'(x,z)=d(x,z)$ and $d'(x,y)=1$. Hence (X,d') satisfies the third property.

Since (X,d') satisfies all three properties, (X,d') is a metric space by defintion

$\boxed{C.3}$

(iii) $d'(x,z) \leq d'(x,y) + d'(y,z)$ for all $x,y,z \in X$.

<u>claim:</u> $\min\{1, a\} + \min\{1, b\} \geq \min\{1, a+b\}$, for all $a,b \geq 0$.

<u>Proof:</u>

<u>$a, b \leq 1$:</u>	$LHS = RHS = a+b$. ← redundant but helpful	
<u>$a, b > 1$:</u>	$LHS = 2$,	$RHS = 1$.
<u>$a > 1, b \leq 1$:</u>	$LHS = 1+b$,	$RHS = 1$.
<u>$a \leq 1, b > 1$:</u>	$LHS = a+1$,	$RHS = 1$. □

Therefore,

$$\begin{aligned} & d'(x,z) \\ = \ & \min\{1, d(x,z)\} \\ \leq \ & \min\{1, d(x,y) + d(y,z)\} \quad \text{(since } (X,d) \text{ is a metric space)} \\ \leq \ & \min\{1, d(x,y)\} + \min\{1, d(y,z)\} \quad \text{(by the claim)} \\ = \ & d'(x,y) + d'(y,z). \end{aligned}$$

□

Advanced Mathematical Economics Tutorial 2
Appendix C: C.1 Metric Spaces

Question C.2.
Suppose that (X, d) is a metric space. Let d ' (x, y) = d(x, y)/(1 + d(x, y)).
Prove that (X, d') is also a metric space.
Hint: If b, c ≥ 0 then b/(1 + b) + c/(1 + c) ≥ b/(1 + b + c) + c/(1 + b + c) = (b + c)/(1 + b + c) = 1 − 1/(1 + b + c).

Suppose (X, d) is a metric space.

Let $d'(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$ for all $x, y \in X$.

To prove that (X, d') is also a metric space, $d'(x,y)$ must satisfy:

(i) $d'(x,y) = 0 \iff x = y$,

(ii) $d'(x,y) = d'(y,x)$ for all $x, y \in X$,

(iii) $d'(x,z) \leq d'(x,y) + d'(y,z)$

Since (X, d) is a metric space, $d(x,y) = 0$ ~~when~~ $\iff$ $x = y$.

When $x = y$,

long-winded, but very careful!

$$\Rightarrow d'(x,x) = \frac{0}{1+0} = 0.$$

When $x \neq y$, $d(x,y) \neq 0$,

$$\Rightarrow d'(x,y) = \frac{d(x,y)}{1+d(x,y)} \neq 0$$

$\therefore$ $d'(x,y)$ satisfies ~~the first property~~ (i), where $d'(x,y) = 0 \iff x = y$. ✓

Since (X, d) is a metric space, $d(x,y) = d(y,x)$ for all $x, y \in X$,

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

$$= \frac{d(y,x)}{1+d(y,x)}$$

$$= d'(y,x)$$

$\therefore$ $d'(x,y) = d'(y,x)$ for all $x, y \in X$ ((ii) is satisfied). ✓

Using the hint, if $b, c \geq 0$, then

$$\frac{b}{1+b} + \frac{c}{1+c} \geq \frac{b}{1+b+c} + \frac{c}{1+b+c}$$

$$= \frac{b+c}{1+b+c}$$

$$= 1 - \frac{1}{1+b+c}.$$

Let $b = d(x,y)$ and $c = d(y,z)$,

$$\frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \geq 1 - \frac{1}{1+d(x,y)+d(y,z)}$$

$$d'(x,y) + d'(y,z) \geq 1 - \frac{1}{1+d(x,y)+d(y,z)}$$ ~~since~~

Since (X,d) is a metric space,

$$d(x,z) \leq d(x,y) + d(y,z)$$

$$\therefore \frac{1}{1+d(x,y)+d(y,z)} \leq \frac{1}{1+d(x,z)}$$

$$1 - \frac{1}{1+d(x,y)+d(y,z)} \geq 1 - \frac{1}{1+d(x,z)}$$

$$\therefore d'(x,y) + d'(y,z) \geq 1 - \frac{1}{1+d(x,y)+d(y,z)} \geq 1 - \frac{1}{1+d(x,z)}$$

$$\Rightarrow \quad d'(x,y) + d'(y,z) \geq 1 - \frac{1}{1+d(x,z)}$$

$$\cancel{d'(x,y)+d'(y,z) \geq} = \frac{1+d(x,z)-1}{1+d(x,z)}$$

$$\cancel{d'(x,y)+d'(y,z) \geq} = \frac{d(x,z)}{1+d(x,z)}$$

$$\cancel{d'(x,y)+d'(y,z) \geq} = d'(x,z)$$

$$d'(x,z) \leq d'(x,y) + d'(y,z)$$ [(iii) is satisfied]

$\therefore$ Since all three condition of $d'(x,y)$ are satisfied, (X, d') is also a metric space. $\square$

Advanced Mathematical Economics Tutorial 2
Appendix C: C.1 Metric Spaces

Question C.3.
Suppose that (X, d) is a metric space. Let d ' (x, y) = min {1, d(x, y)}.
Prove that (X, d') is also a metric space.

Suppose (X, d) is a metric space,

Let $d'(x,y) = \min\{1, d(x,y)\}$

To prove that (X, d') is also a metric space, $d'(x,y)$ must satisfy:

(i) $d'(x,y) = 0 \iff x = y$,

(ii) $d'(x,y) = d'(y,x)$ for all $x, y \in X$,

(iii) $d'(x,z) \leq d'(x,y) + d'(y,z)$

is this (i) ?

$\leq$ $<$

~~d'(x,y)~~ ~~when~~ ~~d(x,y)~~

when $0 \geq d(x,y) > 1$, ← not relevant?

$d'(x,y) = d(x,y)$

~~when d(x,y) ≠ X,~~

~~d'(x,y) = X~~

$\therefore$ when $0 \geq d(x,y) > 1$, $d'(x,y)$ ~~is a metric~~ satisfies all three condition and (X, d') is a metric space.

big picture: 3 properties need to hold for all $x, y \in X$

when $d(x,y) \geq 1$, $d'(x,y) = 1$.

Since the first condition has already been proved above,

the second and third condition shall be proved below,

Since $d(x,y)$ satisfies the conditions ((X, d) is a metric space),

$\Rightarrow d(x,y) = d(y,x)$,

$\therefore$ when $d(y,x) \geq 1$, $d'(y,x) = 1$

$= d'(x,y)$

$\therefore$ The second condition is met.

$\boxed{C.3}$ For the following 3 cases,

1. $d(x,y) \geq 1$, $0 \leq d(y,z) < 1$, $0 \leq d(x,z) < 1$

2. $0 \leq d(x,y) < 1$, $d(y,z) \geq 1$, $0 \leq d(x,z) < 1$

3. $d(x,y) \geq 1$, $d(y,z) \geq 1$, $d(x,z) \geq 1$

$d'(x,z) \cancel{\geq} < d'(x,y) + d'(y,z)$, which satisfies (iii).

For the following 2 cases,

(a) $d(x,y) \geq 1$, $0 \leq d(y,z) < 1$, $d(x,z) \geq 1$

(b) $0 \leq d(x,y) < 1$, $d(y,z) \geq 1$, $d(x,z) \geq 1$

$d'(x,z) \leq d'(x,y) + d'(y,z)$, which satisfies (iii).

For the case where $0 \leq d(x,y) < 1$, $0 \leq d(y,z) < 1$, $d(x,z) \geq 1$,

subject to : $d(x,y) + d(y,z) \geq d(x,z)$,

$$d'(x,y) + d'(y,z) \geq d(x,z) \geq 1$$

$$\Rightarrow d'(x,y) + d'(y,z) \geq 1$$

$$\Rightarrow d'(x,y) + d'(y,z) \geq d'(x,z) \quad \left[\begin{array}{l}\text{since } d'(x,z) = 1 \\ \text{when } d(x,z) \geq 1\end{array}\right]$$

$\therefore$ (iii) is met for this case.

$\therefore$ (X, d') is also a metric space. $\square$

(Since (i), (ii) and (iii) are met for the distance function).

②

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C2) Suppose that (X, d) is a metric space. Let $d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$

Prove that (X, d') is also a metric space.

Hint: If $b, c \geq 0$ then $\frac{b}{(1+b)} + \frac{c}{(1+c)} \geq \frac{b}{(1+b+c)} + \frac{c}{(1+b+c)} = \frac{(b+c)}{(1+b+c)} = 1 - \frac{1}{1+b+c}$

assumption

By definition, (X, d') is a metric space if X is a non-empty set and the distance metric d' satisfies 3 assumptions:

(i) $d'(x,y) = 0 \iff x = y$. Since (X, d) is a metric space, we know that:

a bit confusing

$$d(x,y) = 0 \iff x = y.$$

$$\Rightarrow d'(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{0}{1+0} = 0 \iff x = y$$

so $d'(x,y)$ satisfies this assumption.

(ii) $d'(x,y) = d'(y,x)$ for all $x, y \in X$. Since we know

$$d(x,y) = d(y,x) \quad \text{for all } x, y \in X$$

$$\Rightarrow \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} \quad \text{for all } x, y \in X$$

$$\iff d'(x,y) = d'(y,x) \quad \text{for all } x, y \in X$$

and thus d' satisfies this assumption. ✓

"triangle inequality"

(iii) $d'(x,z) \leq d'(x,y) + d'(y,z)$ for all $x, y, z \in X$. We know that

$$d(x,z) \leq d(x,y) + d(y,z)$$

Big jump!

$$\Rightarrow \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \geq \frac{d(x,y) + d(y,z)}{1 + d(x,y) + d(y,z)}$$

$$\iff d'(x,y) + d'(y,z) \geq \frac{d(x,y) + d(y,z)}{1 + d(x,y) + d(y,z)} \geq \frac{d(x,z)}{1+d(x,z)}$$ ✗

$$\left(\text{since } d(x,z) \leq d(x,y) + d(y,z) \text{ and } \frac{n}{1+n} \geq \frac{n-m}{1+n-m}\right)$$

$$\iff d'(x,y) + d'(y,z) \geq d'(x,z)$$

thus this assumption is satisfied.

∴ all 3 assumptions are satisfied, therefore (X, d') is a metric space.

(3) Suppose (X,d) is a metric space. Let $d'(x,y) = \min\{1, d(x,y)\}$.
Prove (X, d') is also a metric space.

The distance metric d' must satisfy 3 assumptions:

(i) $d'(x,y) = 0 \Leftrightarrow x = y$.

~~$d(x,y) = 0 \Leftrightarrow x = y$ (since (X,d) is a metric space)~~

confusing

~~$\Rightarrow d'(x,y) = \min\{1, 0\} = 0 \Leftrightarrow x = y$~~

so d' satisfies assumption 1.

(ii) $d'(x,y) = d'(y,x)$ for all $x, y \in X$.

~~$d(x,y) = d(y,x)$ (since (X,d) is a metric space)~~

$\Rightarrow d'(x,y) = \min\{1, d(x,y)\} = \min\{1, d(y,x)\} = d'(y,x)$

so d' satisfies assumption 2.

$d'(x,y) = 0$
$\Leftrightarrow d(x,y) = 0$
$\Leftrightarrow x = y$

(iii) $d'(x,y) + d'(y,z) \geq d'(x,z)$.

~~$d(x,y) + d(y,z) \geq d(x,z)$ (since (X,d) is a metric space)~~

~~$\Rightarrow d'(x,y) + d'(y,z) = \min\{1, d(x,y)\} + \min\{1, d(y,z)\}$~~

$= d(x,y) + d(y,z) \geq d(x,z) \geq \min\{1, d(x,z)\} = d'(x,z)$

case 1: $\boxed{\text{when } d(x,y), d(y,z) \leq 1}$ (hard to read)

$= 1 + d(y,z) \geq 1 \geq \min\{1, d(x,z)\} = d'(x,z)$
when $d(x,y) \geq 1$, $d(y,z) \leq 1$,

$= d(x,y) + 1 \geq 1 \geq \min\{1, d(x,z)\} = d'(x,z)$
when $d(x,y) \leq 1$, $d(y,z) \geq 1$,

$= 1 + 1 \geq 1 \geq \min\{1, d(x,z)\} = d'(x,z)$
when $d(x,y), d(y,z) \geq 1$.

therefore,
$$d'(x,y) + d'(y,z) \geq d'(x,z)$$
and this assumption is satisfied.

thorough but long-winded.

$\therefore$ All 3 assumptions are satisfied, therefore (X, d') is a metric space.