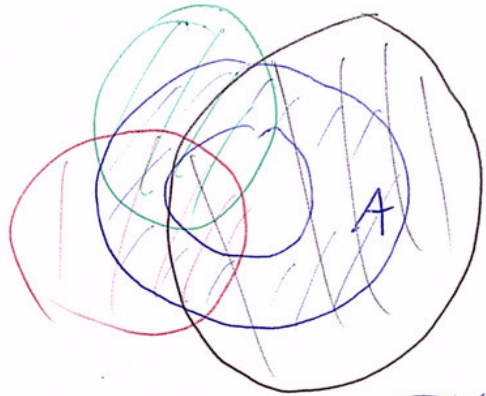
Back to compact sets

<u>Def</u> A <u>cover</u> of a set A is a collection of sets $\mathcal{C}$ such that $A \subseteq \bigcup_{C \in \mathcal{C}} C$.



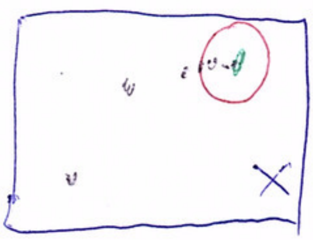
cover —

<u>Def</u> Let $\mathcal{C}$ be a cover A. ~~Thus~~ $\mathcal{C}'$ is a <u>subcover</u> of A if $\mathcal{C}' \subseteq \mathcal{C}$ and $\mathcal{C}'$ is a cover A.

<u>Def</u> Let (X, d) be a metric space. An <u>open cover</u> of A is a cover of A consisting only of open sets in (X, d).

<u>Lemma</u> Let (X, d) be a metric space. If x_n has no convergent subsequence, then for all $x \in X$, there exists $r(x) > 0$ such that $N_{r(x)}(x)$ contains only finitely many x_n.

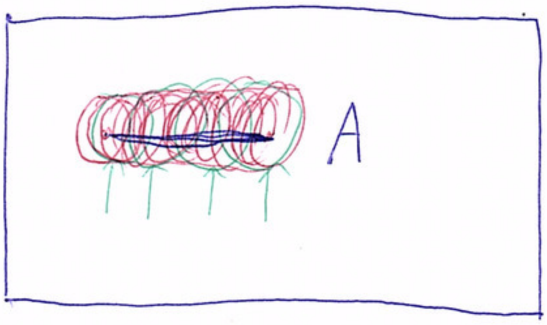
eg:



contains infinitely many points from x_n.

~~Proof~~

<u>Theorem</u> Let (X, d) be metric space. $A \subseteq X$ is compact if and only if every open cover of A has a finite subcover. ← Heine-Borel definition of compactness



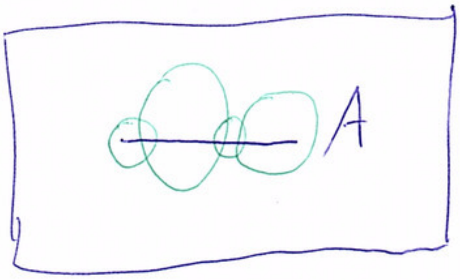
— infinite open cover of A (infinite number of sets)

— a finite subcover of —

<u>Proof</u>: compact ~~⇒~~ open cover has finite subcover

Suppose any open cover of A has a finite subcover, and for the sake of contradiction assume $x_n \in A$ has no convergent subsequence.

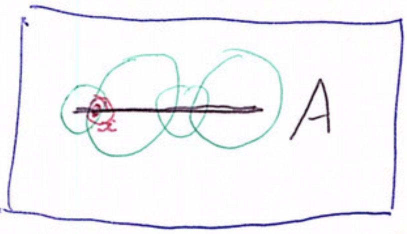
Consider the open cover $\mathcal{G} = \{N_{r(x)}(x) : x \in A\}$ ← By the Lemma of A. Notice each set in $\mathcal{G}$ contains only finitely many x_n. By the condition, $\mathcal{G}$ has a finite subcover $\mathcal{D}$. At least one set $D \in \mathcal{D}$ must have infinitely many x_n, contradicting the definition of $\mathcal{G}$.



$=$ finite subcover D each with a finite number of x_n

finite sub-cover $\Leftarrow$ compact

Conversely, suppose any sequence in A has a convergent subsequence whose limit lies in A. Let $\mathcal{U}$ be any open cover and $f(x) = \sup\{r : N_r(x) \subseteq U$ for some $U \in \mathcal{U}\}$.

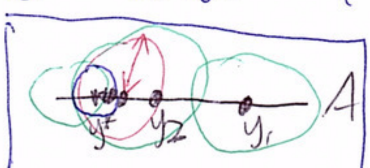


— $\mathcal{U}$

Note that $f(x) > 0$ for all $x \in A$.
Let $r^* = \inf_{x \in A} f(x)$.

infimum (like "minimum")
e.g. $\inf(0,1] = 0$.

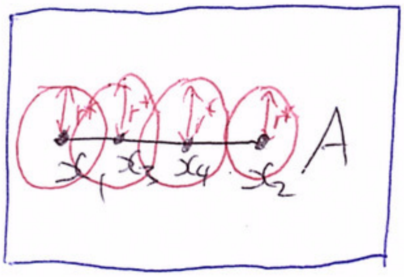
Step one: $r^* > 0$. Suppose (for the sake of contradiction) that $r^* = 0$. There would have to be a sequence $x_n \in A$ such that that $f(x_n) \to 0$. Let $y_n \to y^*$ be a convergent subsequence of x_n. Now $f(y^*) > 0$ and $f(y_n) > f(y^*) - d(y_n, y^*)$



○ around y^*

So $f(y_n) \to f(y^*)$. ↯

Step two:



finite cover, consisting of balls of radius r^*.

Skip: only need a finite number of these balls.

step three Each red ball is contained inside a set in $\mathcal{U}$. Take those sets — these form a finite subcover of $\mathcal{U}$. $\square$

Theorem (Heine-Borel)

Consider a Euclidean metric space $(\mathbb{R}^n, d_2)$, and a subset $X \subseteq \mathbb{R}^n$. Then X is closed and bounded iff every open cover of X has a finite subcover.

Proof X is closed and bounded in $(\mathbb{R}^n, d_2)$

$\Leftrightarrow$ X is compact (Bolzano-Weierstrass theorem)

$\Leftrightarrow$ every open cover of X has a finite sub-cover (theorem we just did). $\square$

Theorem (Cantor's intersection theorem)
Let (X,d) be a metric space, and K_n be a sequence of subsets of X. If each K_n is non-empty, compact and nested (i.e. $K_{n+1} \subseteq K_n$) then

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

Proof

"harmless assumption"

Without loss of generality, assume that $K_1 = X$.

Let $U_n = X \setminus K_n$. Each K_n is closed $\Rightarrow$ each U_n is open.

Suppose for the sake of contradiction that

$$\bigcap_{n=1}^{\infty} K_n = \emptyset.$$

$$\Rightarrow \bigcup_{n=1}^{\infty} U_n = X.$$

This means that $\{U_n\}$ is an open cover of X. Therefore $\{U_n\}$ contains a finite subcover, $\mathcal{U}$.

Since $U_n \subseteq U_{n+1}$, it follows ~~$\mathcal{U}$~~

$$\bigcup_{U \in \mathcal{U}} U = U_N = X \text{ for some } N.$$

This contradicts $U_N = X \setminus K_N$ and K_N being non-empty.