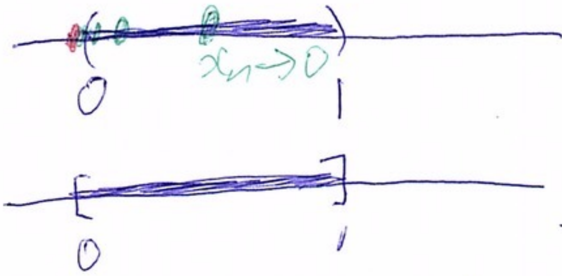
C.9 Compact sets

<u>Def</u> We say a set A in (X,d) is <u>bounded</u> if there is some ball $N_r(x)$ such that $A \subseteq N_r(x)$.

<u>Def</u> Let A be a subset of (X,d). We say that A is <u>compact</u> if for every sequence $a_n \in A$, there exists a convergent subsequence $y_n \to y^*$ such that $y^* \in A$. We say (X,d) is a <u>compact metric space</u> if X is compact in (X,d).

not compact



is compact
- but need a theorem.

<u>Theorem (Bolzano-Weierstrass)</u>
Let A be a subset of $(\mathbb{R}^n, d_2)$. Then A is compact iff A is both closed and bounded.

<u>Proof</u> ① compact $\Rightarrow$ closed & bounded. true in any metric space

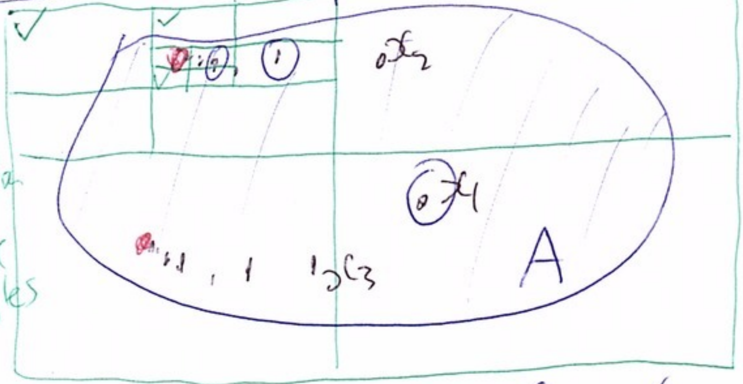
Suppose A is compact, and for the sake of contradiction, unbounded.

Then pick any $a_0 \in A$. For every n, ~~therefore~~ $A \not\subseteq N_n(a_0)$, so we can pick $a_n \in A \setminus N_n(a_0)$. This sequence — and all of its subsequences — are unbounded, and hence not convergent. This contradicts the condition that A is compact. We conclude A is bounded.

Let $a_n \in A$ be any convergent sequence with $a_n \to x^*$. To show A is closed, we need to prove that $x^* \in A$. Since A is compact, a_n has some convergent subsequence $b_n \to b^*$ s.t. $b^* \in A$. But these two sequences (a_n & b_n) have to converge to the same thing, i.e. $x^* = b^*$. We conclude $x^* \in A$.

<u>closed & ~~compact~~ bounded $\Rightarrow$ compact</u> ← only true in Euclidean space

used bounded assumption to make rectangles



Pick y_n as any point from (x_n) that lies inside the n^{th}-zoomed-in rectangle

<u>Claim</u>: y_n is a Cauchy sequence. Since $(\mathbb{R}^n, d_2)$ is complete, y_n is convergent, i.e. $y_n \to y^*$. We need to prove $y^* \in A$. This is true because $y_n \in A$ and A is closed. $\square$

Theorem Suppose $f : X \to Y$ is continuous and surjective ($f(X) = Y$) and that (X, d_X) is a compact metric space. Then (Y, d_Y) is also a compact metric space.

~~Proof~~ e.g. If $X \subseteq \mathbb{R}^{NH}_+$ is the set of feasible allocations, is compact and

$$u(x) = (u_1(x_1), u_2(x_2), \dots, u_H(x_H))$$

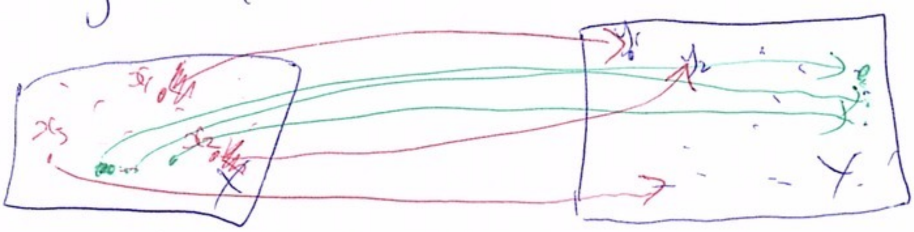
is continuous, then the feasible utility set

$$\mathcal{U} = \{u(x) : x \in X\} = u(X)$$

is a compact set.

Want to prove: y_n has a convergent subsequence

Proof of theorem Let $y_n \in Y$ be any sequence. Since f is surjective, there is a sequence $x_n \in X$ such that $f(x_n) = y_n$. Since X is compact, x_n has a convergent subsequence $x_{n_k} \to x^*$. ($z_k = x_{n_k}$) Since f is continuous, $f(x_{n_k}) \to f(x^*)$. Let $y^* = f(x^*)$. We conclude $y_{n_k} \to y^*$. $\square$



Extreme value theorem

Suppose $f: X \to \mathbb{R}$ is a continuous function ~~between~~ from (X, d) to $(\mathbb{R}, d_2)$.
If X is compact and non-empty, then f has a maximum (and a minimum), i.e.

$$\max_{x \in X} f(x)$$

has a solution.

Proof Let $Y = f(X) \subseteq \mathbb{R}$. By the previous theorem, Y is a compact set in $(\mathbb{R}, d_2)$. By the Bolzano-Weierstrass theorem, Y is closed and bounded in $(\mathbb{R}, d_2)$. Since Y is bounded,

$$\sup_{y \in Y} y \quad \text{is finite.}$$

Since Y is closed, $\left(\sup_{y \in Y} y\right) \in Y$.
So Y has a maximum. $\square$

Blackwell's Lemma Suppose u is increasing, continuous and bounded. Then the Bellman operator ~~contract~~ $F: CB(\mathbb{R}_+) \to CB(\mathbb{R}_+)$ is a contraction of degree β. (according to d_∞)

Proof Pick any $V' \in CB(\mathbb{R}_+)$.

1. $V = F(V')$ exists. Since V' is continuous, the objective:

$$(x, k') \mapsto u(x) + \beta V(k')$$

is continuous. Moreover, the constraint, $\{(x, k') \in \mathbb{R}^2_+ : x + k' \leq k\}$ is compact for all k, by the Bolzano-Weierstrass theorem. By the extreme value theorem, the max exists, so $F(V')(k)$ exists for all k.

2. V is continuous (skipped).

$\Rightarrow$ 3. F is a self-map on $(CB(\mathbb{R}_+), d_\infty)$.

4. We will show F is a contraction of degree β.

Let V_1' and $V_2' \in CB(\mathbb{R}_+)$, with corresponding policies $x_1(k)$ and $x_2(k)$.

Then

$$F(V_1')(k) = u(x_1(k)) + \beta V_1'(k - x_1(k))$$

$$= u(x_1(k)) + \beta V_2'(k - x_1(k)) - \beta V_2'(k - x_1(k)) + \beta V_1'(k - x_1(k))$$

$$\leq \left[\max_{x \in [0,k]} u(x) + \beta V_2'(k-x)\right] + \beta V_1'(k - x_1(k)) - \beta V_2'(k - x_1(k))$$

$$= F(V_2')(k) - \beta V_2'(k - x_1(k)) + \beta V_1'(k - x_1(k))$$

This implies

$$F(V_1')(k) - F(V_2')(k) \leq \beta \left[V_1'(k - x_1(k)) - V_2'(k - x_1(k))\right]$$

$$\leq \beta\, d_\infty(V_1', V_2').$$

Swapping V_1' and V_2' gives

$$F(V_2')(k) - F(V_1')(k) \leq \beta\, d_\infty(V_1', V_2').$$

Combining gives

$$|F(V_1')(k) - F(V_2')(k)| \leq \beta\, d_\infty(V_1', V_2').$$

Since this is true for any $k \in \mathbb{R}_+$, this implies

$$d_\infty(F(V_1'), F(V_2')) \leq \beta\, d_\infty(V_1', V_2').$$

So F is a contraction of degree β. $\square$

Since the Bellman operator is a contraction, we learn that:

- there is a solution ~~that~~ to the Bellman equation (but we already knew this from the principle of optimality)
- there is only one solution to the Bellman equation. (No "wrong" solutions.)
- If we can show $F: A \to A$ on a closed subset A of $CB(\mathbb{R}_+)$, then we can conclude that the ~~sol~~ value function is inside A. e.g. A might be the set of increasing functions, concave functions or functions that have some behaviour of economic interest. (cut-offs.)
- algorithm for calculating the value function.