Theorem C.14 (again!)

Consider (X, d_X) and (Y, d_Y). If (Y, d_Y) is complete, then $(B(X,Y), d_\infty)$ and $(CB(X,Y), d_\infty)$ are complete.

Proof $B(X,Y)$ is complete:

Let $f_n \in B(X,Y)$ be a Cauchy sequence. Pick any $x \in X$. Then $y_n = f_n(x)$ is a Cauchy sequence in (Y, d_Y). Since (Y, d_Y) is complete, $y_n \to y^*$ for some $y^* \in Y$. Define $f^*(x) = y^*$, i.e. $f^*(x) = \lim_{n\to\infty} f_n(x)$.

By continuity of d_Y, we know

$$d_Y(f^*(x), f_n(x)) = \lim_{m\to\infty} d_Y(f_m(x), f_n(x)), \text{ for all } x.$$

$$\leq \lim_{m\to\infty} \sup_{x'\in X} d_Y(f_m(x'), f_n(x')),\text{ "}$$

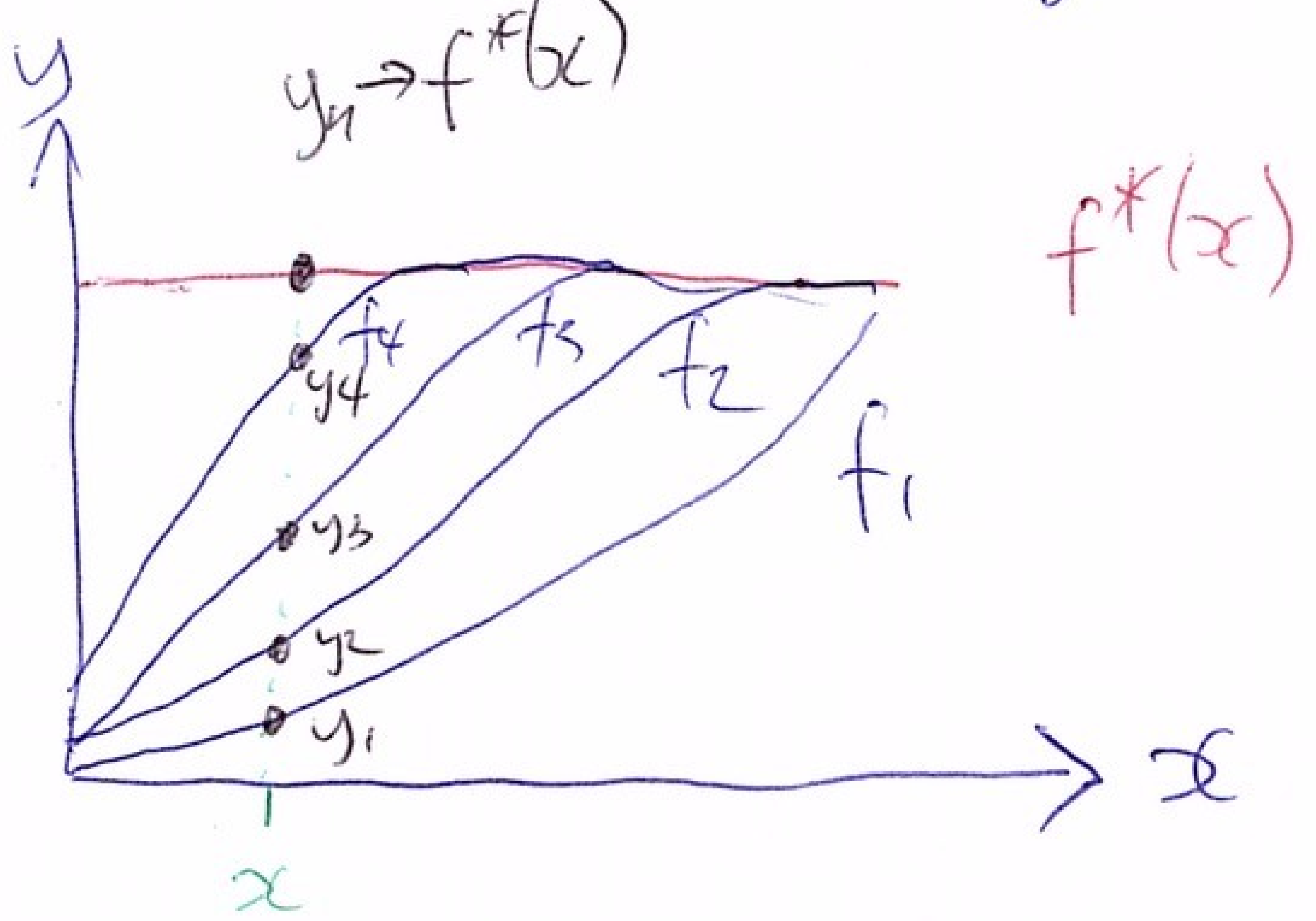
$$= \lim_{m\to\infty} d_\infty(f_m, f_n), \text{ for all } x.$$

So $\sup_{x\in X} d_Y(f^*(x), f_n(x)) \leq$

$$d_\infty(f^*, f_n) \leq \underbrace{\lim_{m\to\infty} d_\infty(f_m, f_n)}_{\text{as } n\to\infty,\ \text{"} \to 0}.$$

Conclusion: $d_\infty(f^*, f_n) \to 0$ which implies $f_n \to f^*$. $\square$

What could have gone wrong?



<u>Back to dynamic programming in section 2.4</u>

$$\pi(p; w) = \max_{x \in \mathbb{R}_+^{N-1}} pf(x) - w \cdot x$$

$$\pi(p; w) = \max_{y \in \mathbb{R}_+} py - c(y; w) \leftarrow \text{Bellman equation}$$

where $$c(y; w) = \min_{x \in \mathbb{R}_+^{N-1}} w \cdot x$$
$$\text{s.t. } f(x) \geq y.$$

<u>Lemma</u> (Principle of Optimality)
The definitions of π agree.

<u>Proof</u> $$\max_{x \in \mathbb{R}_+^{N-1}} pf(x) - w \cdot x$$

point-less! $$= \max_{y \in \mathbb{R}_+,\, x \in \mathbb{R}_+^{N-1}} pf(x) - w \cdot x$$

$$= \max_{\substack{y \in \mathbb{R}_+, \\ x \in \mathbb{R}^{N-1}_+}} \quad p f(x) - w \cdot x$$

$$\text{s.t.} \quad f(x) = y$$

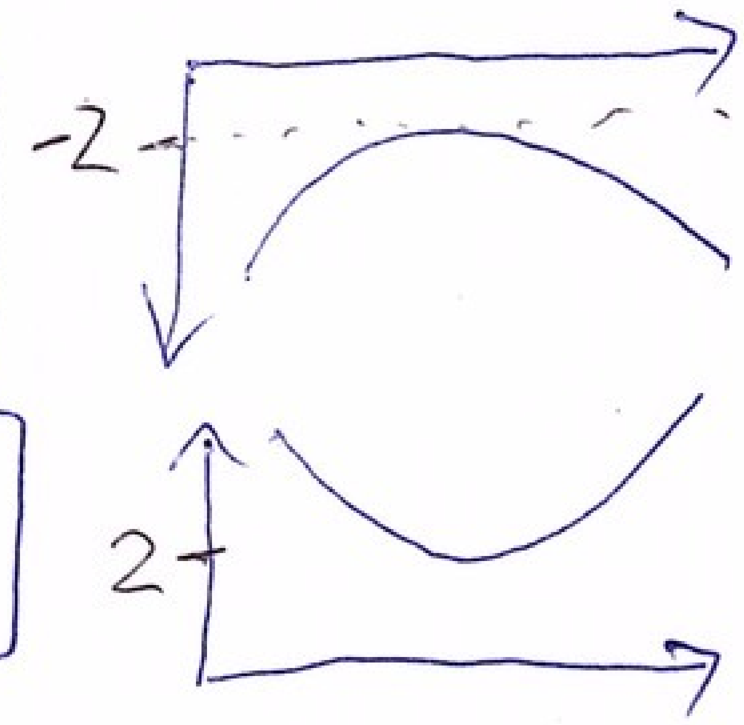
$\leftarrow$ slack ("pointless") constraint

$$= \max_{y \in \mathbb{R}_+} \left[\max_{x \in \mathbb{R}^{N-1}_+} \; p f(x) - w \cdot x \quad \text{s.t.} \quad f(x) = y \right]$$

$$= \max_{y \in \mathbb{R}_+} \left[\max_{x \in \mathbb{R}^{N-1}_+} \; p y - w \cdot x \quad \text{s.t.} \quad f(x) = y \right]$$

$$= \max_{y \in \mathbb{R}_+} \; p y + \left[\max_{x \in \mathbb{R}^{N-1}_+} \; - w \cdot x \quad \text{s.t.} \quad f(x) = y \right]$$

$$= \max_{y \in \mathbb{R}_+} \; p y - \left[\min_{x \in \mathbb{R}^{N-1}_+} \; w \cdot x \quad \text{s.t.} \quad f(x) = y \right]$$



$$= \max_{y \in \mathbb{R}_+} \; p y - c(y; w). \qquad \square$$

<u>Theorem</u> If $y(p; w)$ is an optimal supply policy, then

$$p = \left[\frac{\partial c(y; w)}{\partial y} \right]_{y = y(p; w)}$$

Appendix G: Infinite Horizon Dynamic Programming

Cake of size k.

$$V_t(k) = \max_{(x_s)_{s=1}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} u(x_s)$$

$$\text{s.t.} \sum_{s=t}^{\infty} x_s = k.$$

Bellman equation:

$$V_t(k) = \max_{x_t,\, k_{t+1} \geq 0} u(x_t) + \beta V_{t+1}(k_{t+1})$$

$$\text{s.t. } x_t + k_{t+1} = k.$$

Since V_t does not depend on t:

$$V(k) = \max_{x,\, k' \geq 0} u(x) + \beta V(k')$$

$\boxed{'}$ means "tommorrow"

recursive Bellman equation

$$\text{s.t. } x + k' = k.$$

Theorem (Principle of Optimality)

Suppose there is a solution to the cake eating problem. Then the value function V_0 is a solution to the Bellman equation.

Proof: $$V_0(k) = \max_{(x_s)_{s=0}^{\infty}} \sum_{s=0}^{\infty} \beta^s u(x_s) \text{ s.t. } \sum_{s=0}^{\infty} x_s = k$$

$$= \max_{x_0,\, k_1,\, (x_s)_{s=1}^{\infty}} \sum_{s=0}^{\infty} \beta^s u(x_s)$$

$$\text{s.t. } x_0 + k_1 = k \quad \text{and} \quad \sum_{s=1}^{\infty} x_s = k_1$$

$$= \max_{\substack{x_0, k_1 \\ \text{s.t. } x_0 + k_1 = k}} \left[\begin{array}{l} \max_{(x_s)_{s=1}^{\infty}} \sum_{s=0}^{\infty} \beta^s u(x_s) \\ \text{s.t. } \sum_{s=1}^{\infty} x_s = k_1 \end{array} \right]$$

$$= \max_{\substack{x_0, k_1 \\ \text{s.t. } x_0 + k_1 = k}} \left[\begin{array}{l} \max_{(x_s)_{s=1}^{\infty}} u(x_0) + \sum_{s=1}^{\infty} \beta^s u(x_s) \\ \text{s.t. } \sum_{s=1}^{\infty} x_s = k_1 \end{array} \right]$$

$$= \max_{\substack{x_0, k_1 \\ \text{s.t. } x_0 + k_1 = k}} u(x_0) + \beta \left[\begin{array}{l} \max_{x_{s=1}^{\infty}} \sum_{s=1}^{\infty} \beta^{s-1} u(x_s) \\ \text{s.t. } \sum_{s=1}^{\infty} x_s = k_1 \end{array} \right]$$

$$= \max_{\substack{x_0, k_1 \\ \text{s.t. } x_0 + k_1 = k}} u(x_0) + \beta V_0(k_1)$$

$$= \max_{\substack{x, k' \\ \text{s.t. } x + k' = k}} u(x) + \beta V_0(k') \qquad \square$$

Theorem (Banach fixed point theorem)

Let (X, d) be a complete metric space. If $f: X \to X$ is a contraction of degree a, then

(i) f has a unique fixed point x^*,

(ii) Given any $x_0 \in X$, then the sequence defined by $x_{n+1} = f(x_n)$ converges to x^*,

(iii) $d(x_n, x^*) \leq \frac{a^n}{1-a} d(x_0, x_1)$ for all n.

Proof (i) uniqueness: Suppose both x^* and x^{**} are fixed points of f, where $x^* \neq x^{**}$. Since they are fixed points, $x^* = f(x^*)$ and $x^{**} = f(x^{**})$. Applying the contraction property,

$$d(f(x^*), f(x^{**})) \leq a\, d(x^*, x^{**})$$

$$\Rightarrow d(x^*, x^{**}) \leq a\, d(x^*, x^{**})$$

$$\Rightarrow d(x^*, x^{**}) < d(x^*, x^{**}) \quad ↯$$

(i) existence: We first show x_n is a Cauchy sequence. If we repeatedly apply the contraction property, we get:

$$d(f(x_0), f(x_m)) \leq a\, d(x_0, x_m)$$

$$d(f(f(x_0)), f(f(x_m))) \leq \cancel{a^2 d(x_0, x_m)}\ a\, d(f(x_0), f(x_m)) \leq a^2 d(x_0, x_m)$$

$$(*) \quad d(f^n(x_0), f^n(x_m)) \leq a^n d(x_0, x_m).$$

This implies triangle inequality

$$d(x_0, x_m) \le \underbrace{d(x_0,x_1)}_{} + d(x_1,x_2) + \cdots + d(x_{m-1},x_m)$$

$n=1, m=1$ $\quad$ $n=m-1, m=1$

$$< d(x_0,x_1) + a\,d(x_0,x_1) + a^2 d(x_0,x_1) + \cdots$$

$$= d(x_0,x_1)\left[1 + a + a^2 + \cdots\right]$$

$$= \cancel{d(x_0,x_1)}\ \frac{1}{1-a}\, d(x_0,x_1).$$

Rewriting the formula (*) above:

$$d(x_n, x_{n+m}) \le a^n d(x_0, x_m)$$

$$\le a^n \left\{ \frac{1}{1-a} d(x_0,x_1) \right\} \text{ for all } n, m$$

$$\Rightarrow \quad d(x_n, x_m) \le \frac{a^N}{1-a} d(x_0,x_1) \quad \text{for all } n, m \ge N. \qquad (**)$$

So x_n is a Cauchy sequence.

Since (X, d_X) is a complete metric space, $x_n \to x^*$ for some $x^* \in X$. Let $y_n = f(x_n)$. Since f is continuous, and $x_n \to x^*$ we conclude $y_n \to f(x^*)$. But y_n is a subsequence of x_n (since $y_n = x_{n+1}$), we conclude $y_n \to x^*$. This means $x^* = f(x^*)$. So x^* is a fixed point of f.

(iii) <u>Approximation bound</u>: ~~By continuity of~~

$$d(x_n, x^*) = \lim_{m\to\infty} d(x_n, x_m) \le \frac{a^n}{1-a} d(x_0, x_1). \qquad \square$$

since d is continuous $\qquad$ from (**)

Back to the cake-eating problem

"tomorrow's value function"

Bellman operator:

$$F(V')(k) = \max_{\substack{x, k' \geq 0 \\ \text{s.t. } x + k' = k}} u(x) + \beta V'(k').$$

$$F : B(\mathbb{R}_+) \to B(\mathbb{R}_+)$$

Our goal is to use Banach's fixed point theorem to find a fixed point of F. Algorithm:

1. Start with a guess V_0.
2. $V_{n+1} = F(V_n)$
3. Stop at V_N, where N is chosen to ensure $d_\infty(V_N, V)$ is small enough.