Completeness

e.g. $((0,1], d_1)$. $x_n = \frac{1}{n}$ "wants" to converge to 0. But $0 \notin (0,1]$! So is not convergent.

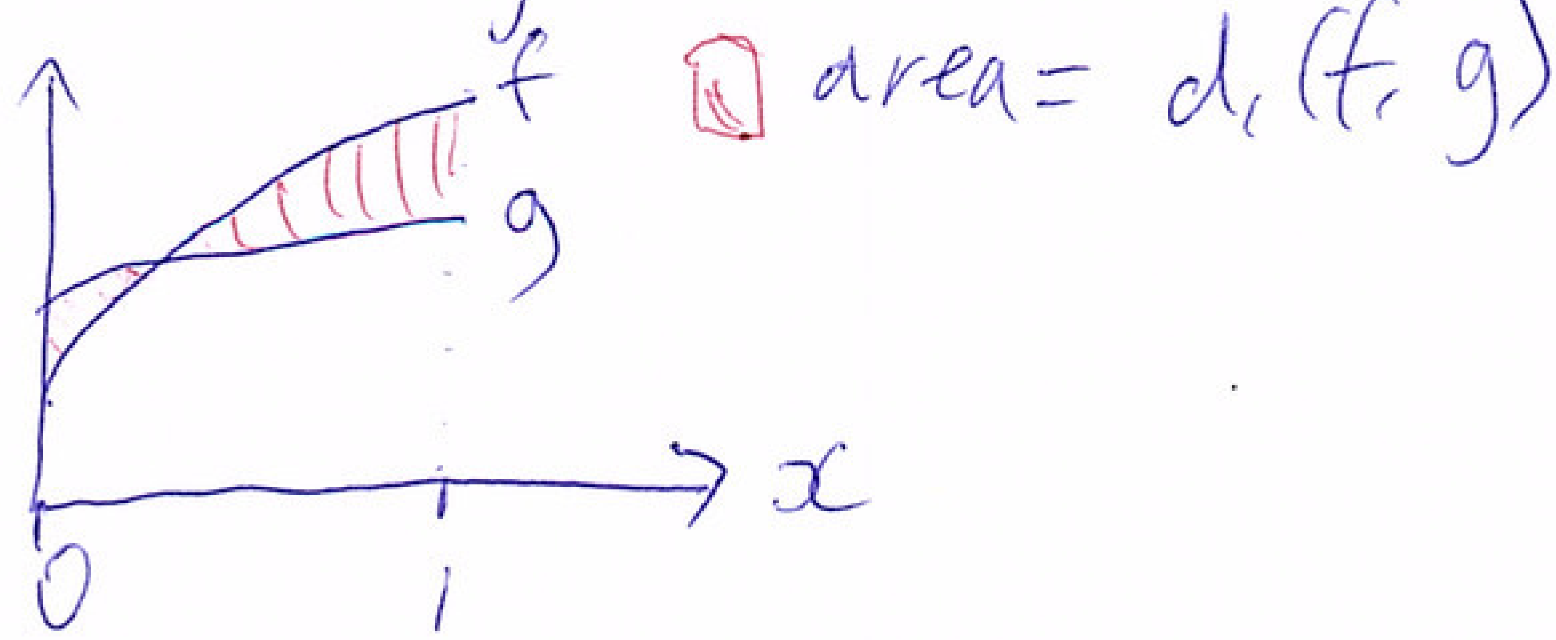
Def Let (X,d) be a metric space. A sequence $x_n \in X$ is called a Cauchy sequence if for every radius $r>0$, there exists some N such that

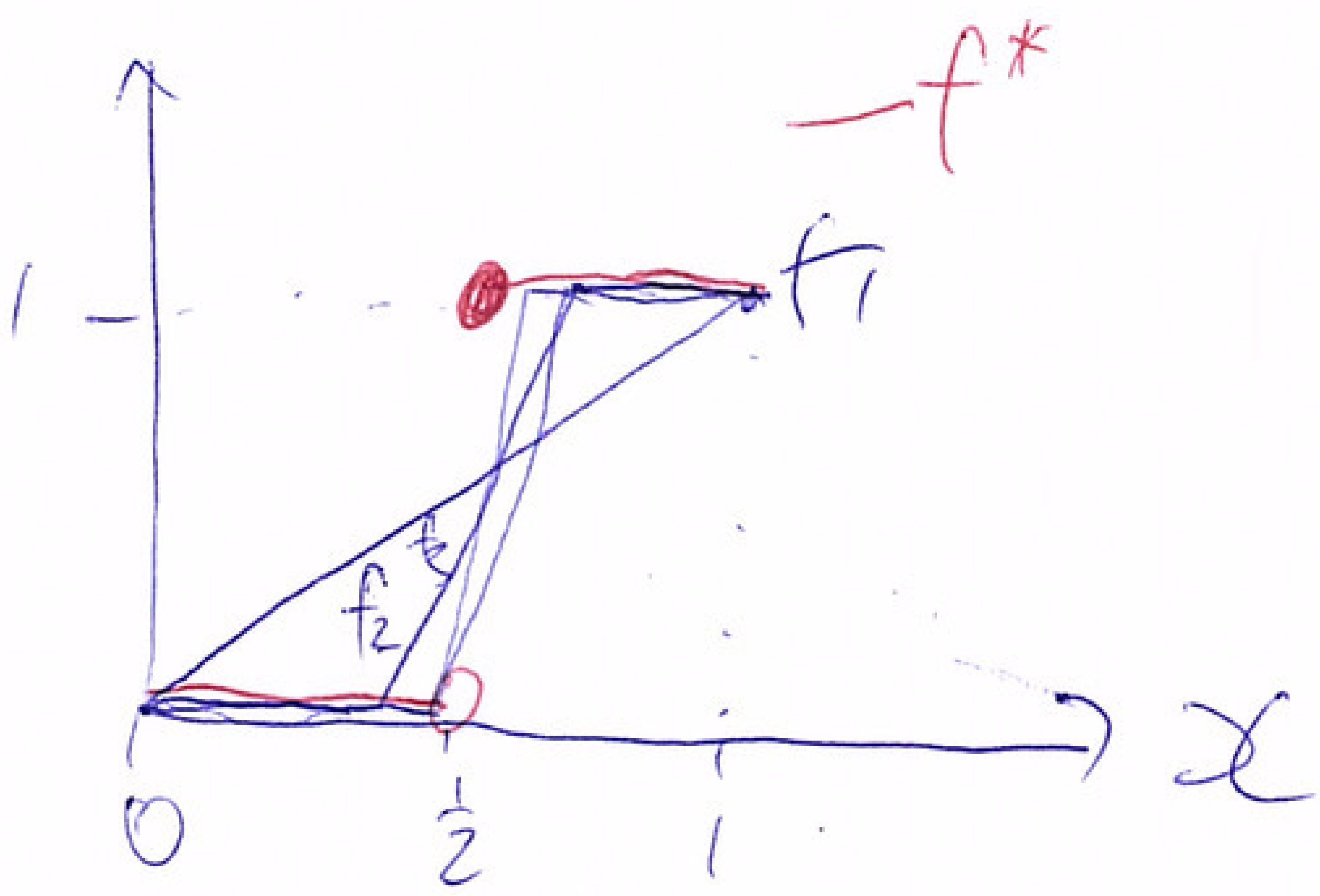
$$d(x_n, x_m) < r \quad \text{for all } n, m > N.$$

Def A metric (X,d) is complete if every Cauchy sequence is convergent.

eg: * $(\mathbb{R}, d_2)$ is complete.

* $((0,1], d_1)$ is NOT complete because $x_n = \frac{1}{n}$ is a non-convergent Cauchy sequence
* $(\mathbb{Q}, d_2)$ is NOT complete.
  eg. $x_1 = 3$, $x_2 = 3.1$, $x_3 = 3.14$, $x_4 = 3.141$, etc.
  x_n "wants" to converge to π.
  But $\pi \notin \mathbb{Q}$. So x_n is non-convergent Cauchy sequence.
* $(CB([0,1]), d_1)$ where $d_1(f,g) = \int_0^1 |f(x) - g(x)|\,dx$ is NOT complete.





Does $f_n \to f^*$?
It is true that $d_1(f_n, f_{n+1}) \to 0$.
This is a Cauchy sequence.
But $f^* \notin CB([0,1])$
since f^* is discontinuous.

So f_n is a ~~discontinuous~~ non-convergent Cauchy sequence.

<u>Theorem</u> Let (X, d) be any metric space.
If $x_n \in X$ is a convergent sequence, then x_n is a Cauchy sequence.

<u>Theorem</u> If $x_n \in X$ is a Cauchy sequence, and $y_n \to y^*$ is a convergent subsequence, then $x_n \to y^*$.

<u>Theorem</u> If $x_n \in X$ is a Cauchy sequence, then x_n is bounded.

<u>Theorem</u> If $x_n \in X$ is a Cauchy sequence, and y_n is a subsequence of x_n, then y_n is a Cauchy sequence.

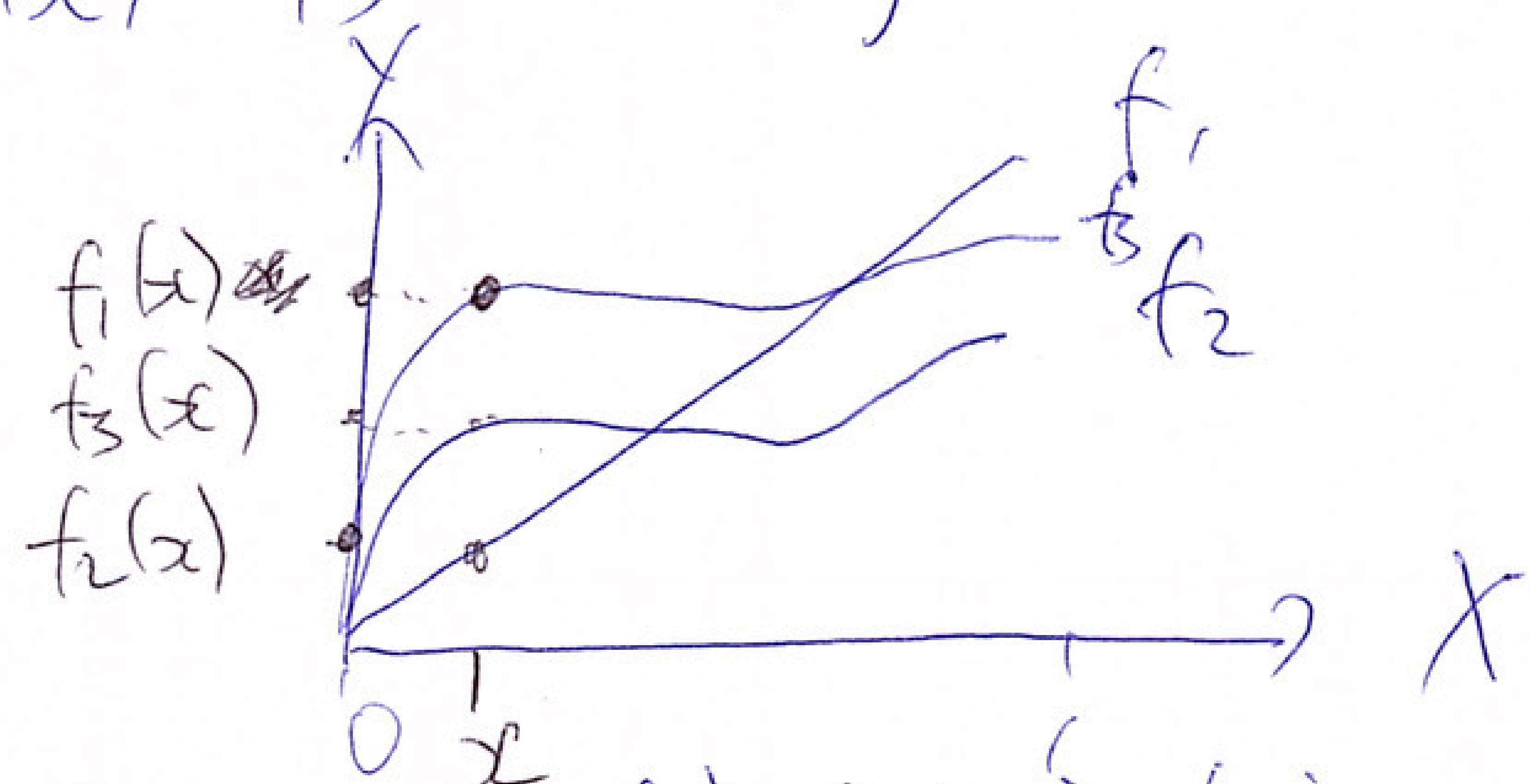
<u>Theorem</u> $(\mathbb{R}, d_2)$ is a complete metric space.

<u>Theorem</u> Let (X, d_X) and (Y, d_Y) be metric spaces. If (Y, d_Y) is complete, then $(B(X,Y), d_\infty)$ and $(CB(X,Y), d_\infty)$ are complete.

<u>Proof</u> <u>$(B(X,Y), d_\infty)$:</u> Let $f_n \in B(X,Y)$ be a Cauchy sequence. We want to prove that f_n is convergent.

Since f_n is Cauchy, it follows that $f_n(x) \in Y$ is Cauchy for all $x \in X$. Since (Y, d_Y) is complete, $f_n(x)$ is convergent.

Let $f^*(x) = \lim_{n\to\infty} f_n(x)$.
It remains to prove that $f_n \to f^*$.



$$d(f^*(x), f_n(x)) = \lim_{m\to\infty} d(f_m(x), f_n(x)) \quad \text{(since } d \text{ is continuous)}$$
$$\leq \lim_{m\to\infty} d_\infty(f_m, f_n)$$

for all $x \in X$.

Need to come back

C.8 Fixed Points

Suppose $f: X \to X$. ← self-map

Def $x^* \in X$ is a *fixed point* of f if $f(x^*) = x^*$.

Def Let (X, d_X) and (Y, d_Y) be metric spaces, and let $a > 0$. A function $f: X \to Y$ is *Lipschitz continuous of degree a* if

$$d_Y(f(x), f(x')) \le a\, d_X(x, x').$$

Def Let (X, d) be a metric space. The self-map $f: X \to X$ is a *contraction* if it is Lipschitz continuous of degree $a < 1$.