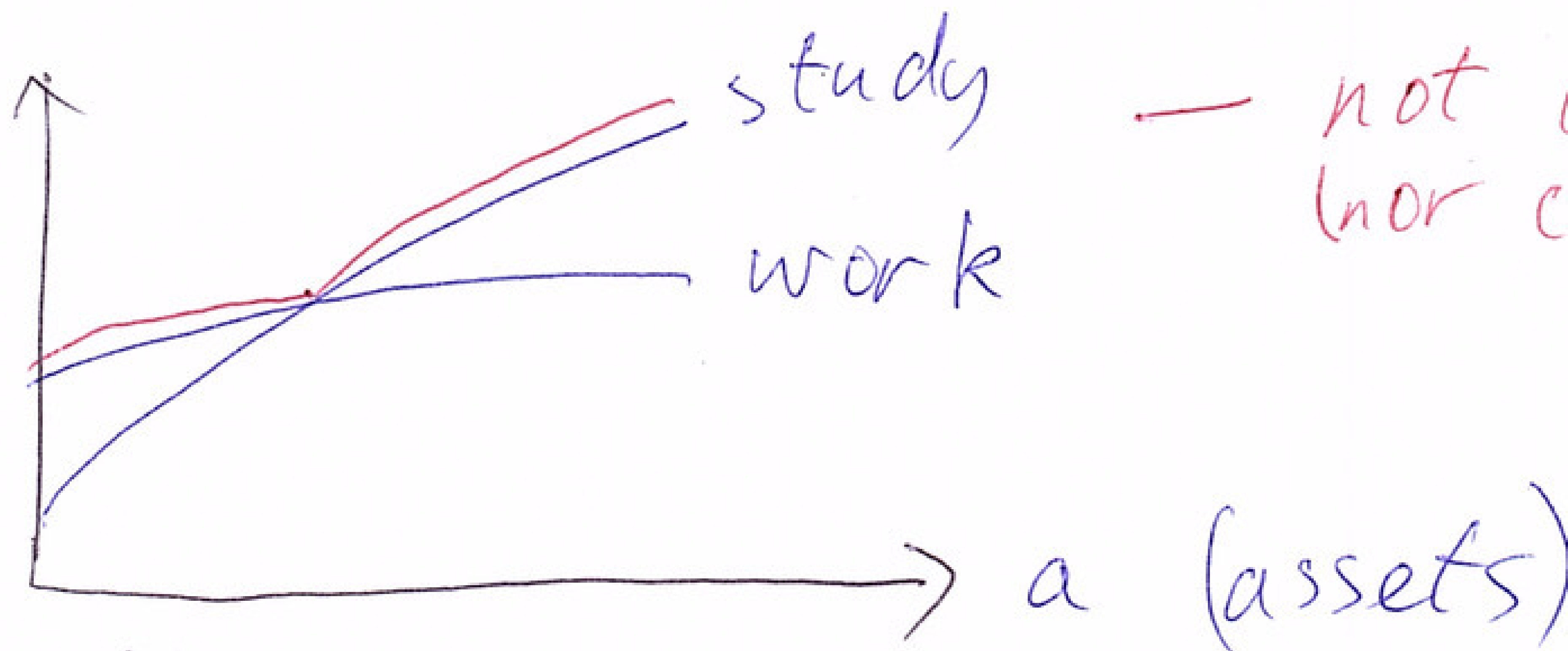
__Theorem__ Suppose $V(a) = \max_b v(a,b)$.

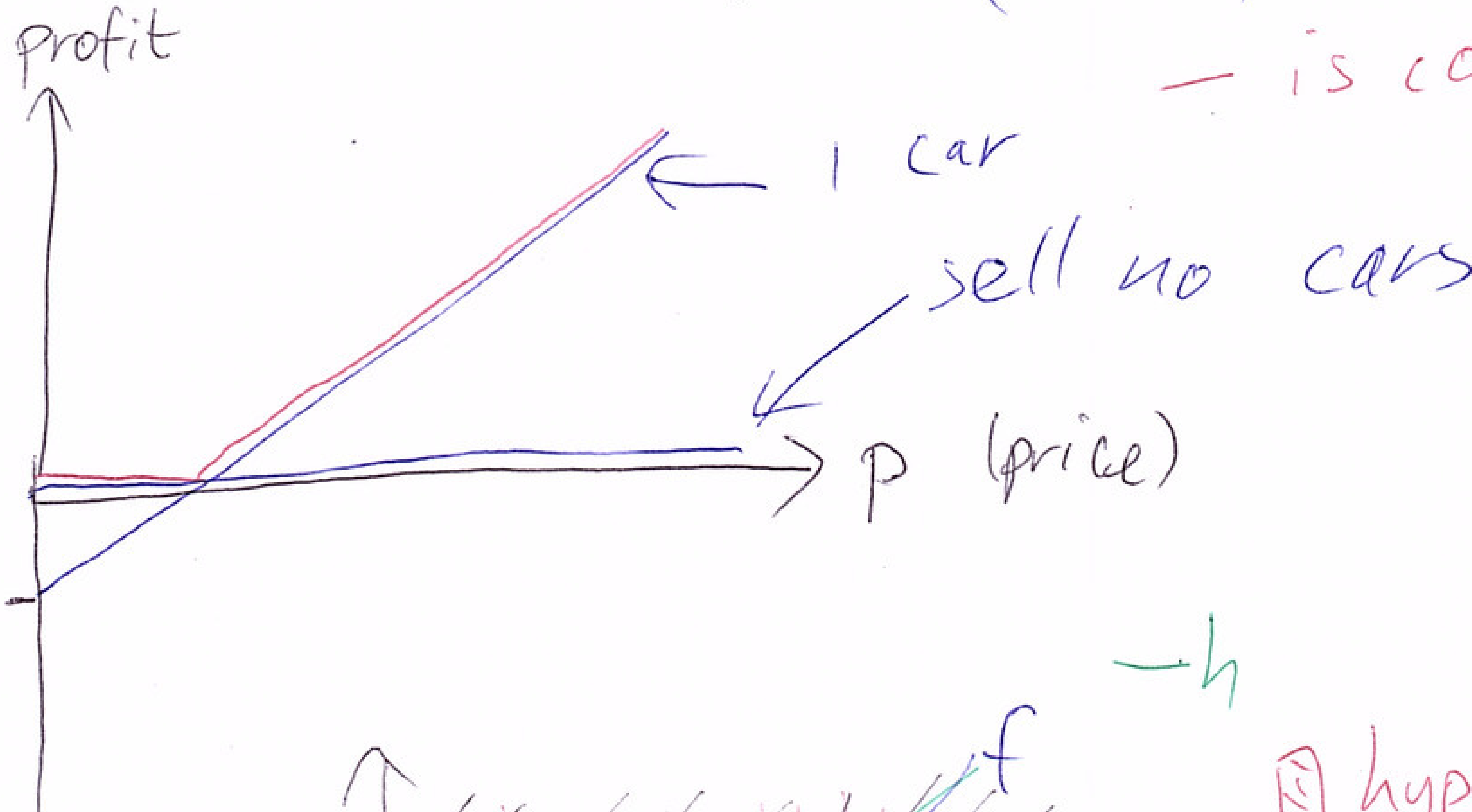
Suppose each $v(\cdot, b)$ is a convex function.

$g(a) = v(a, b)$.

Then V is a convex function.



— not concave (nor convex)



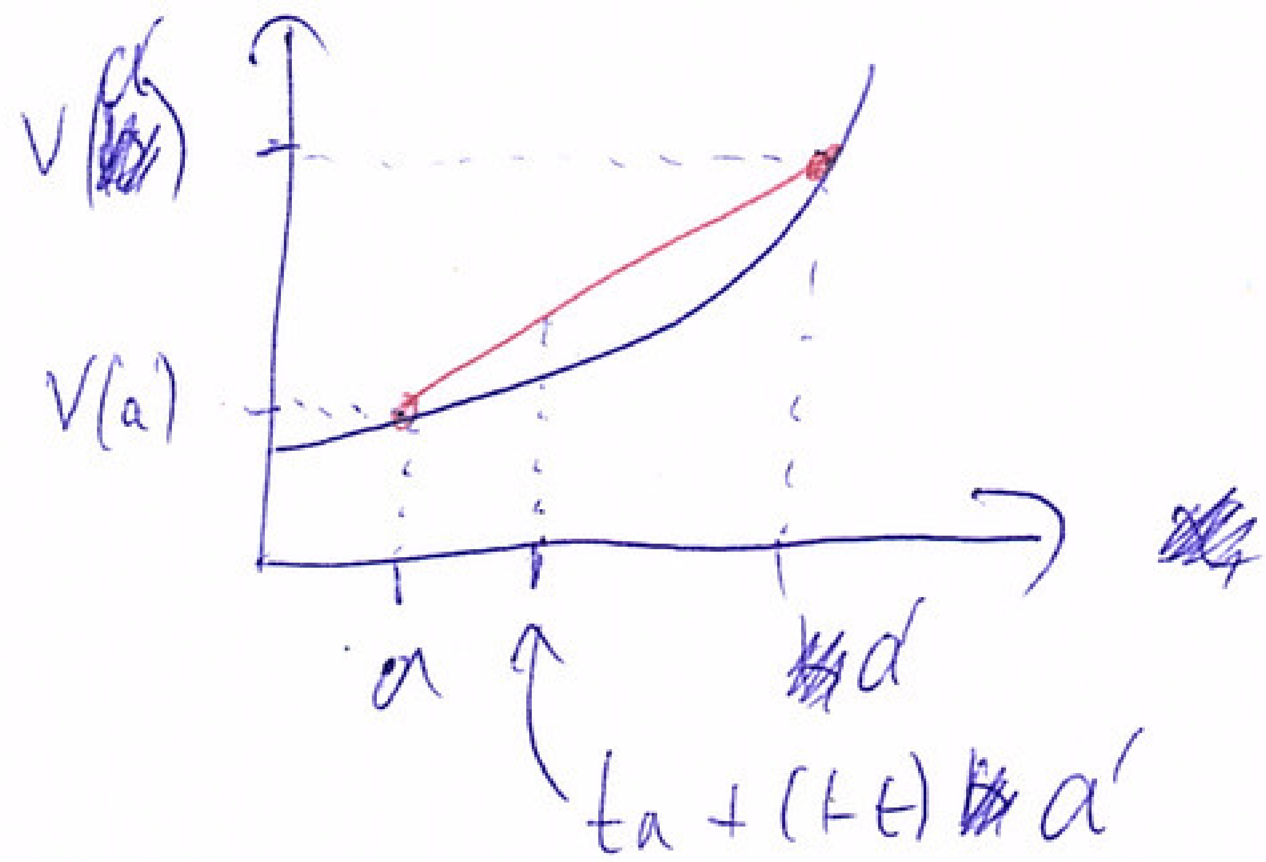
— is convex

both convex sets

hyper(h)

<u>Proof</u> We would like to prove that

$$t V(a) + (1-t) V(a') \geq V(ta + (1-t)a')$$



$$t V(a) + (1-t) V(a')$$

$$\cancel{= t v(a, b(a)) + (1-t) v(}$$

$$= t v(a, b(a)) + (1-t) v(a', b(a'))$$

since b(a) is optimal at a

$$\geq t v(a, b(ta + (1-t)a')) + (1-t) v(a', b(a'))$$

$$\geq t v(a, b(ta + (1-t)a')) + (1-t) v(a', b(ta + (1-t)a'))$$

$$\geq v(ta + (1-t)a', b(ta + (1-t)a'))$$

since $v(\cdot, b)$ is convex for all b

$$= V(ta + (1-t)a'). \qquad \square$$

Theorem For every production function f (need not be concave), π is convex, and if π is smooth, then

$$\frac{\partial y(p;w)}{\partial p} \geq 0 \quad \text{and} \quad \frac{\partial x_i(p;w)}{\partial w_i} \leq 0.$$

Proof: Recall $\pi(p;w) = \max_x pf(x) - w \cdot x$.

This is an upper envelope of linear functions,

$$(p;w) \mapsto pf(x) - w \cdot x,$$

one function for each x.

Since linear functions are convex, the previous theorem implies that π is a convex function.

If π is smooth, $\frac{\partial^2 \pi(p;w)}{\partial p^2} \geq 0$ and $\frac{\partial^2 \pi(p;w)}{\partial w_i^2} \geq 0$. By the envelope theorem, we proved

$$\frac{\partial \pi(p;w)}{\partial p} = y(p;w); \quad \frac{\partial \pi(p;w)}{\partial w_i} = -x_i(p;w)$$

$$\Rightarrow \underbrace{\frac{\partial^2 \pi(p;w)}{\partial p^2}}_{\geq 0} = \frac{\partial y(p;w)}{\partial p}; \quad \underbrace{\frac{\partial^2 \pi(p;w)}{\partial w_i^2}}_{\geq 0} = -\frac{\partial x_i(p;w)}{\partial w_i}.$$

$\square$

2.4 Cost functions & dynamic programming

C G A T

blue eyes: C C C A A T C G G

green eyes: C C C A G A T T G G

Profit function:

$$\pi(p;w) = \max_{x \in \mathbb{R}^{N-1}_+} pf(x) - w.x$$

Bellman equation

$$\underline{\pi(p;w)} = \max_{y \in \mathbb{R}} py - \underline{c(y;w)}$$

value functions

where $c(y;w) = \min_{x \in \mathbb{R}^{N-1}_+} \cancel{\max}\ \cancel{p f(x)}\ w.x$

s.t. $f(x) \geq y$.

Lemma Principle of Optimality

("The Bellman equation is true")