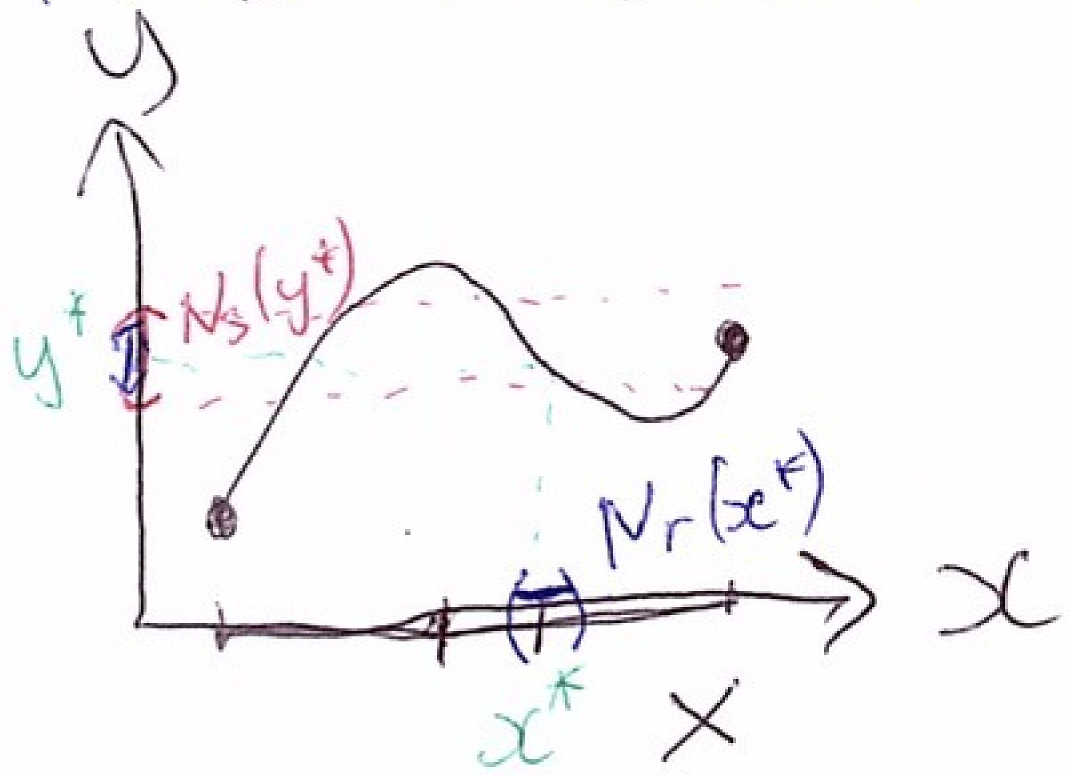
Theorem Let $f: X \to Y$ between (X, d_X) and (Y, d_Y). Pick any $x^* \in X$ and let $\underbrace{y^* = f(x^*)}_{\text{"image of } x^*\text{"}}$. Then f is continuous at x^* iff for every open ball $N_s(y^*)$, there exists an open ball $N_r(x^*)$ such that $f(N_r(x^*)) \subseteq N_s(y^*)$.



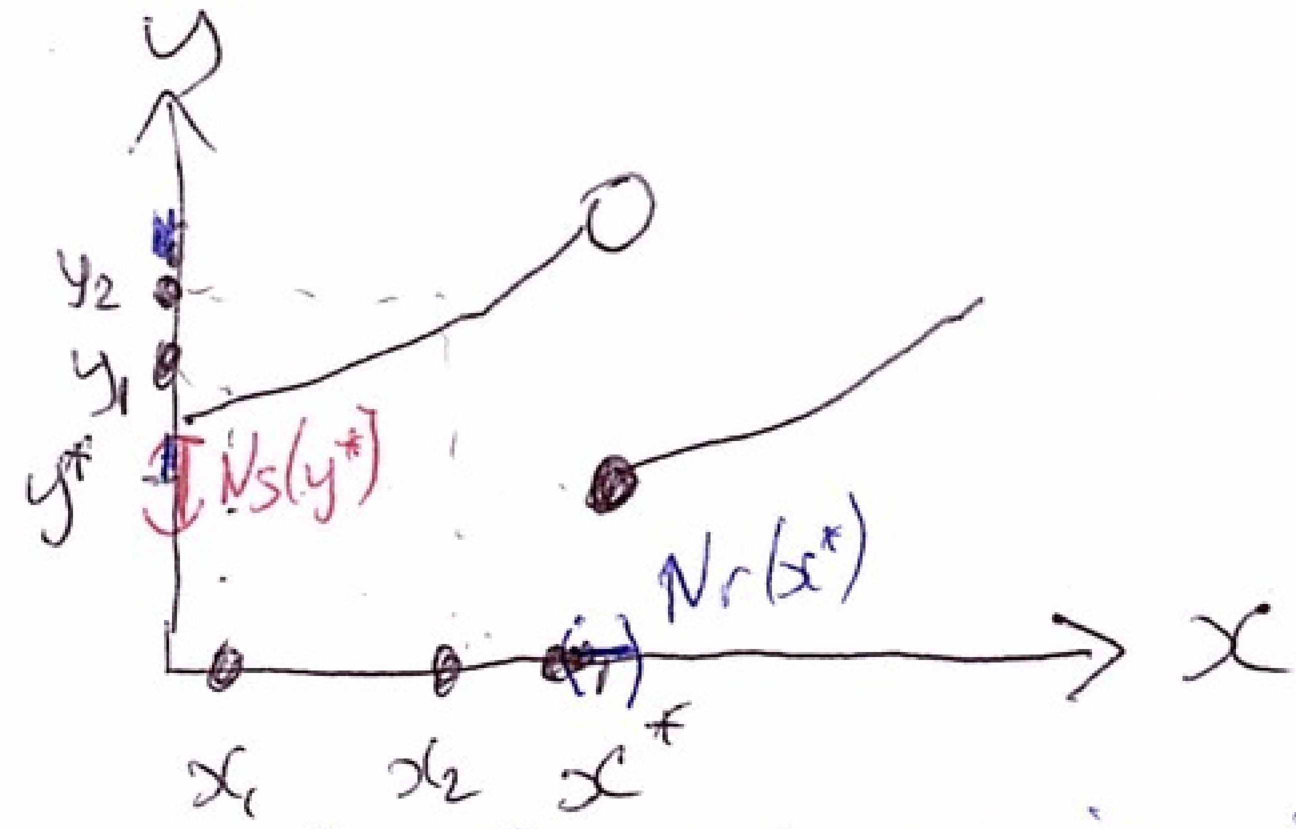
Proof A: f is continuous at x^*

B: for every $N_s(y^*)$, there exists some $N_r(x^*)$ s.t. $f(N_r(x^*)) \subseteq N_s(y^*)$

Instead of proving $A \Rightarrow B$ and $B \Rightarrow A$, I will prove $\neg A \Rightarrow \neg B$ and $\neg B \Rightarrow \neg A$.

$\neg B$: for some $N_s(y^*)$, there is no $N_r(x^*)$ s.t. $f(N_r(x^*)) \subseteq N_s(y^*)$.

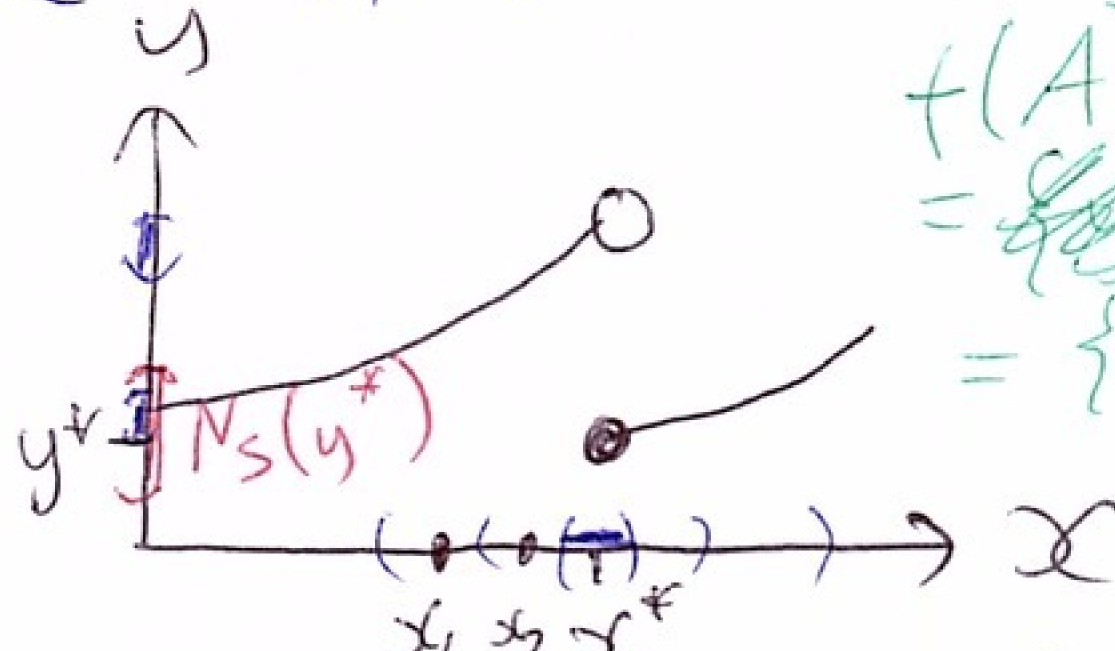
$\neg A \Rightarrow \neg B$: Suppose $x_n \to x^*$, but $y_n = f(x_n) \not\to y^*$.
We will find a ball $N_s(y^*)$ such that every ball $N_r(x^*)$ has $f(N_r(x^*)) \not\subseteq N_s(y^*)$.



Since $y_n \not\to y^*$, there is some $N_s(y^*)$ such that no tail of y_n lies (entirely) in $N_s(y^*)$.
Since every ball $N_r(x^*)$ contains a tail of x_n, it follows that for every $r>0$,
$f(N_r(x^*)) \not\subseteq N_s(y^*)$.

$\neg B \Rightarrow \neg A$: Suppose that for some ball $N_s(y^*)$, there is no ball $N_r(x^*)$ such that $f(N_r(x^*)) \subseteq N_s(y^*)$. We will construct a sequence $x_n \to x^*$ such that $y_n = f(x_n) \not\to y^*$.

$f(A) = \{f(x) : x \in A\}$



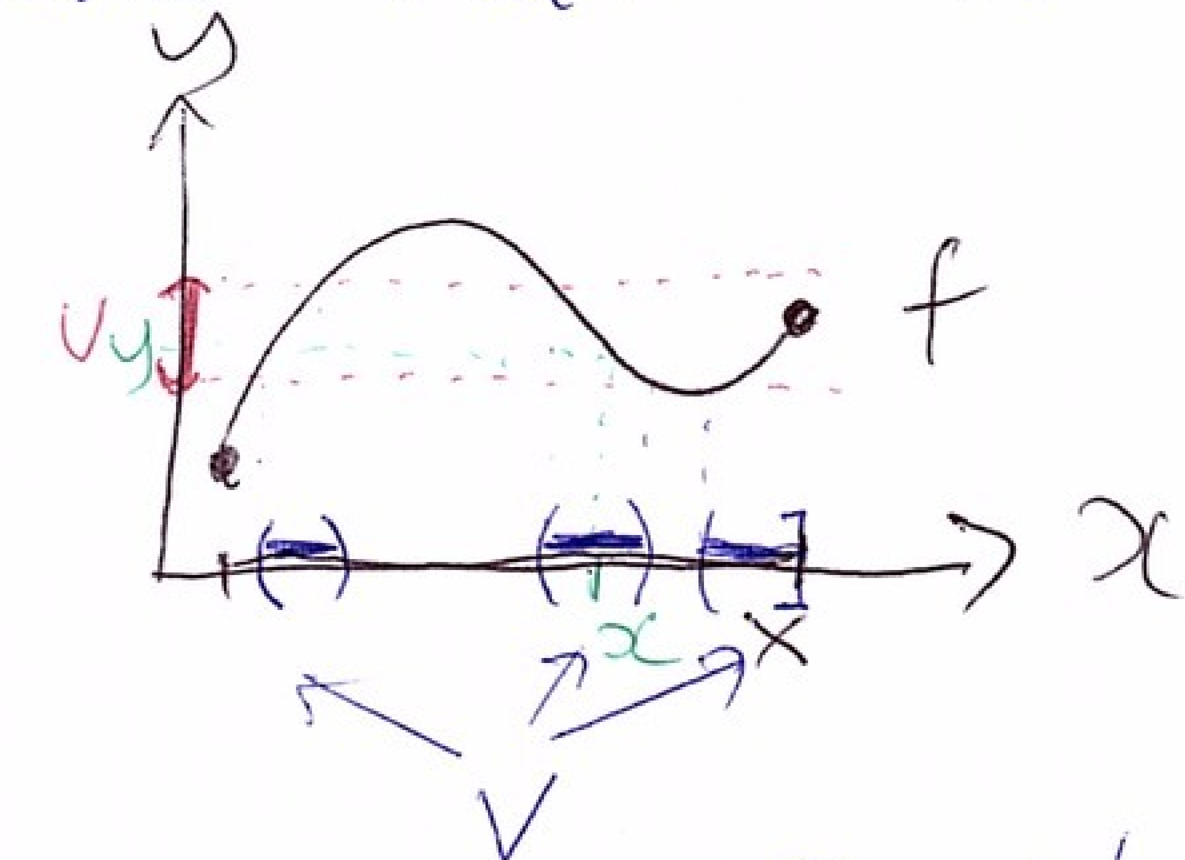
For every n, there exists some $x_n \in N_{\frac{1}{n}}(x^*)$ such that $f(x_n) \notin N_s(y^*)$.
Now $x_n \to x^*$ (since $d(x_n, x^*) < \frac{1}{n}$), but $f(x_n) \not\to y^*$.

$\square$

Theorem ~~Let~~ Let $f: X \to Y$ be a function from (X, d_X) to (Y, d_Y). Then f is continuous iff $f^{-1}(U)$ is an open set for ~~any~~ all open sets $U \subseteq Y$.

"inverse image" $\{x \in X : f(x) \in U\}$

Proof: $\Rightarrow$ Suppose f is continuous. Let U be any open set in (Y, d_Y), and let $V = f^{-1}(U)$. We need to prove that V is an open set in (X, d_X).

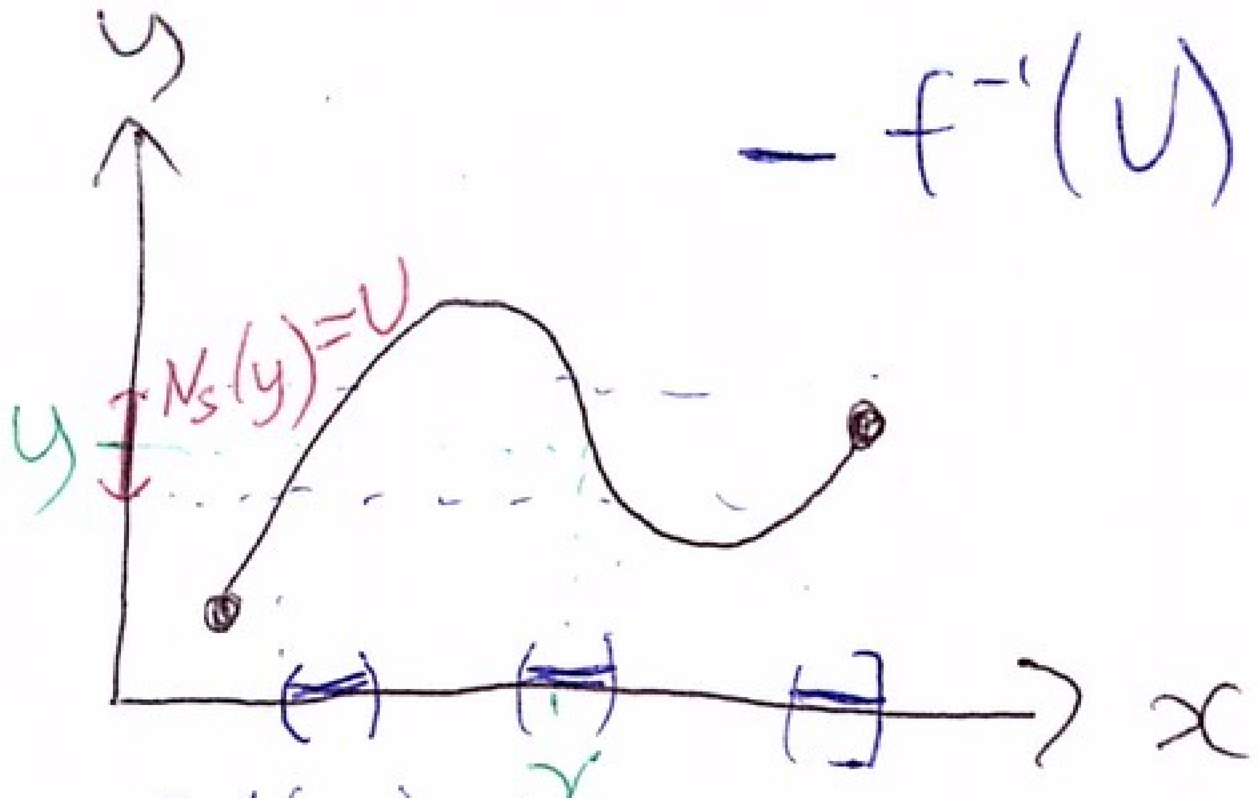


Pick any $x \in V$, and let $y = f(x)$. Since U is an open set, there is ~~a~~ a ball $N_s(y) \subseteq U$. By the previous theorem, there is ~~an~~ a ball $N_r(x)$ such that $f(N_r(x)) \subseteq N_s(y) \subseteq U$. It follows that

$$N_r(x) \subseteq f^{-1}(f(N_r(x))) \subseteq f^{-1}(U) = V.$$

So V is an open set.

$\Leftarrow$ Conversely, suppose that for all open sets $U \subseteq Y$, the set $f^{-1}(U)$ is open. We will show that f is continuous at every $x \in X$. Pick any $x \in X$ and let $y = f(x)$. ~~and~~ Pick any $U = N_s(y)$. So $f^{-1}(U)$ is open.



Since $f^{-1}(U)$ is an open set, and $x \in f^{-1}(U)$, there is a ball $N_r(x)$ such that $N_r(x) \subseteq f^{-1}(U)$. This implies $f(N_r(x)) \subseteq U = N_\delta(y)$. By previous theorem, f is continuous at x. $\square$