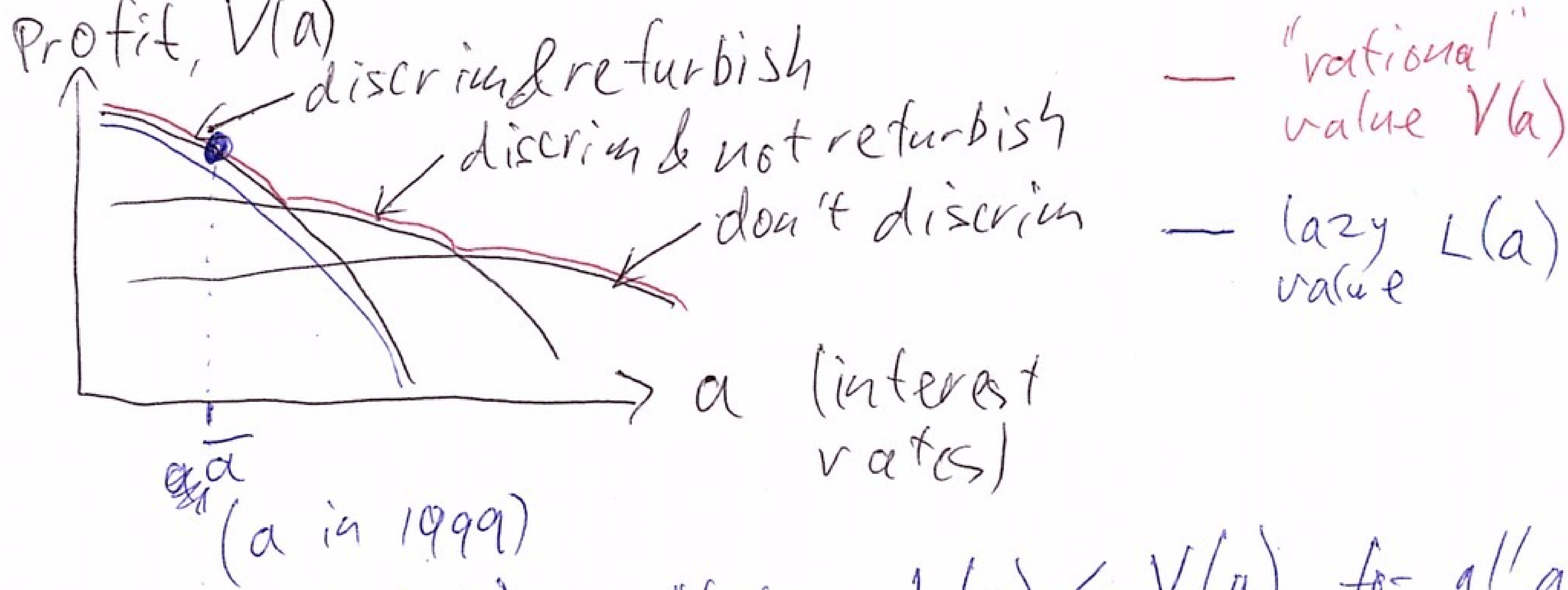
Proof (lazy decision maker proof)

Fix a particular state $\bar{a}$.



$L(a) = v(a, b(\bar{a}))$. Note: $L(a) \leq V(a)$ for all a.

$L(\bar{a}) = V(\bar{a})$. Therefore, $\bar{a}$ solves

$$\min_a V(a) - L(a).$$

FOC: $V'(\bar{a}) - L'(\bar{a}) = 0 \iff V'(\bar{a}) = L'(\bar{a})$.

Finally, $L'(\bar{a}) = v_a(\bar{a}, b(\bar{a}))$. □

Proof ⊛ (chain rule proof)

$V(a) = v(a, b(a))$.

By the chain rule: $V'(a) = v_a(a, b(a)) + \underbrace{v_b(a, b(a))\, b'(a)}_{\text{indirect effect}}$

Since $b(a)$ is an optimal choice, FOC:

$$v_b(a, b(a)) = 0.$$

Therefore, $V'(a) = v_a(a, b(a))$. □

Applying to $\pi(p;w)$,

$$\frac{\partial \pi(p;w)}{\partial p} = \left[\frac{\partial}{\partial p}\left(p f(x) - w \cdot x\right)\right]_{x = x(p;w)}$$

$$= \left[f(x)\right]_{x=x(p;w)}$$

$$= f(x(p;w))$$

$$= y(p;w) \leftarrow \text{output policy.}$$

$$\frac{\partial \pi(p;w)}{\partial w_i} = \left[\frac{\partial}{\partial w_i}\left(p f(x) - w \cdot x\right)\right]_{x=x(p;w)}$$

$$= \left[-x_i\right]_{x=x(p;w)}$$

$$= -x_i(p;w).$$