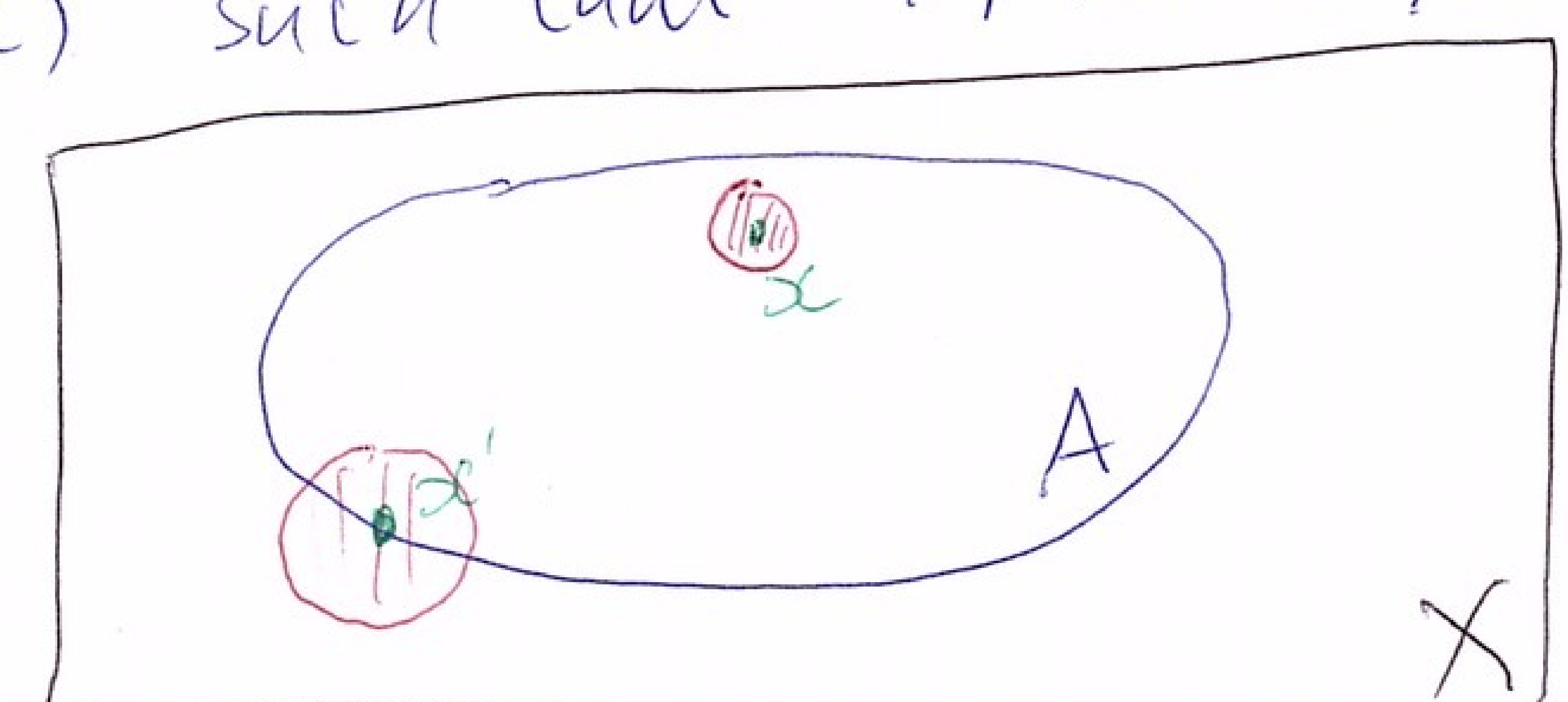
C.5 Open Sets

__Def__ The __open ball__ of radius $r>0$ centred at a point x inside the metric space (X, d) is
$$N_r(x) = \{y \in X : d(x,y) < r\}.$$

~~Def~~

$N_r(x)$ with d_2 (red bit is excluded) $N_r(x)$ with d_1

__Def__ Suppose A is a subset in (X, d). We say that $x \in A$ is an __interior point__ of A if there is an open ball $N_r(x)$ such that $N_r(x) \subseteq A$.



$x \in \text{int}(A)$ but $x' \notin \text{int}(A)$

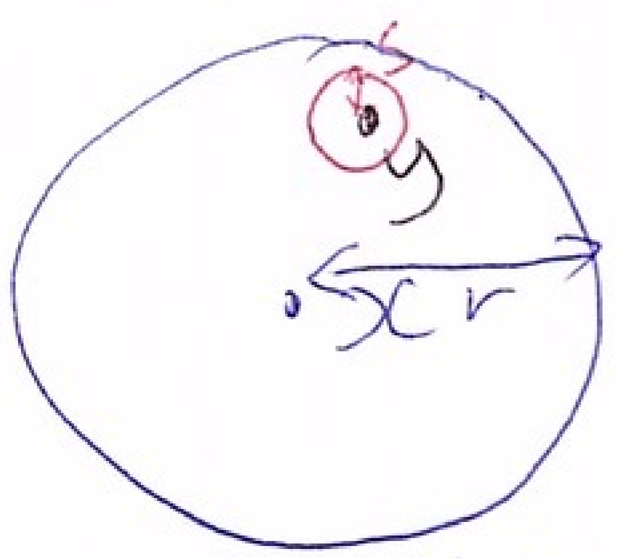
Def The set of interior points of A is called the **interior** of A, denoted $\text{int}(A)$ (or sometimes A°).

Def We say A is an **open set** if $A = \text{int}(A)$.

Def If $x \in A$ and A is an open set, we say that A is an **open neighbourhood** or just **neighbourhood** of x.

e.g.

* open balls are open sets.



eg: $s = r - d(x,y)$

$\Rightarrow N_s(y) \subseteq N_r(x)$.

* $(0,1)$ in $(\mathbb{R}, d_2)$ is an open set.

In fact $(0,1) = N_{\frac{1}{2}}(\frac{1}{2})$.

* X, $\emptyset$ are open sets in (X, d).

eg: $([0,1], d_2)$

$= X$

$A = X$

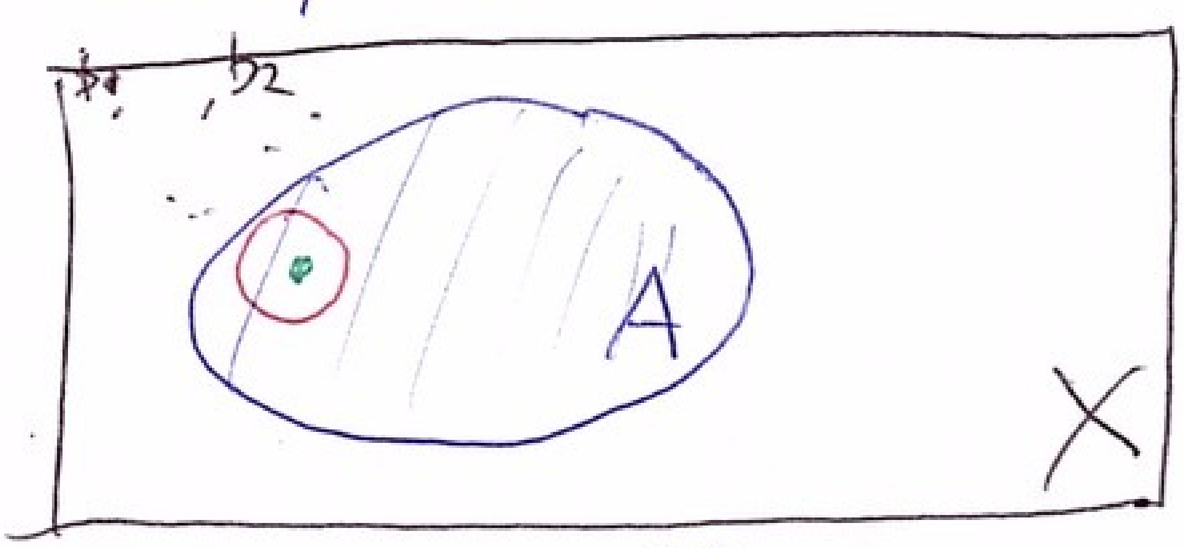
$N_{\frac{1}{2}}(1) = (\frac{1}{2}, 1]$.

So $N_{\frac{1}{2}}(1) \subseteq X$ and $1 \in \text{int}(X)$.

* $[0,1]$ in $(\mathbb{R}, d_2)$ is NOT an open set because $N_r(1) \not\subseteq [0,1]$.

Theorem Let A be a subset in (X,d). Then A is an open set if and only if A contains none of its boundary, i.e. $A \cap \partial A = \emptyset$.

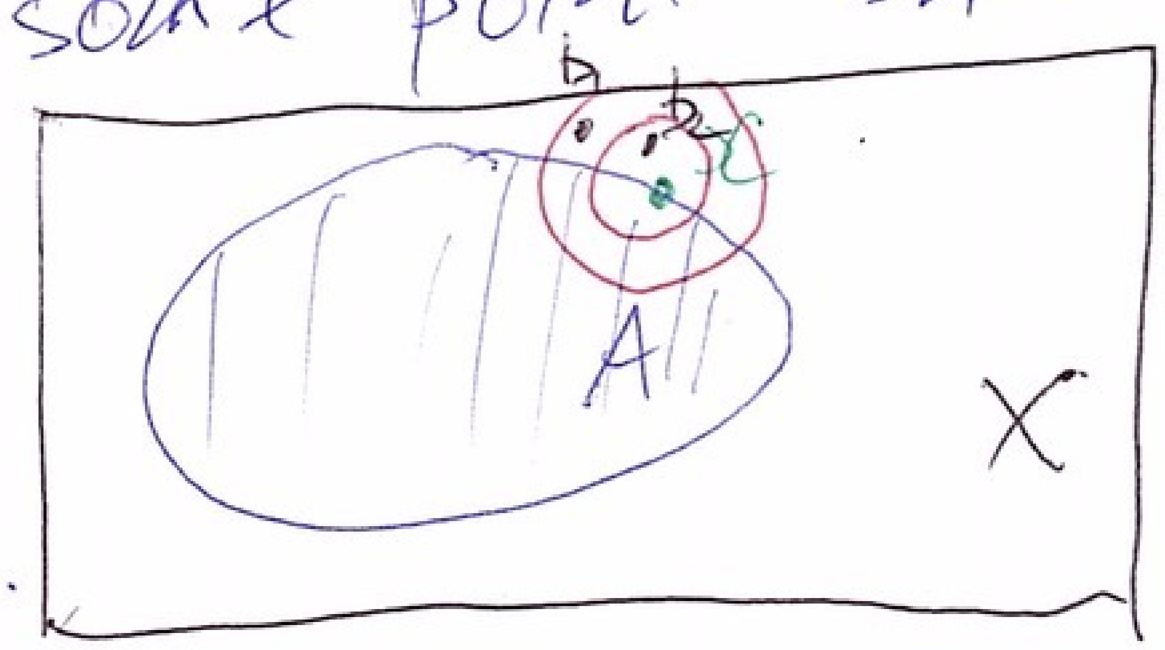
Proof $\Rightarrow$ Consider any point $x \in A$. Since ~~A~~ A is an open set, there some open ball $N_r(x) \subseteq A$. Therefore, no sequence $b_n \notin A$ can converge to x. So $x \notin \partial A$ and hence $A \cap \partial A = \emptyset$.



contrapositive

$\Leftarrow$ Suppose A is not open. That means there some point $x \in A$ such that every open ball $N_r(x) \not\subseteq A$. In particular $N_{\frac{1}{n}}(x) \not\subseteq A$ for all n. So for each n, we can pick some point $b_n \in N_{\frac{1}{n}}(x)$ such that $b_n \notin A$.

Since $d(b_n, x) < \frac{1}{n}$, $b_n \to x$. Therefore $x \in \partial A$.



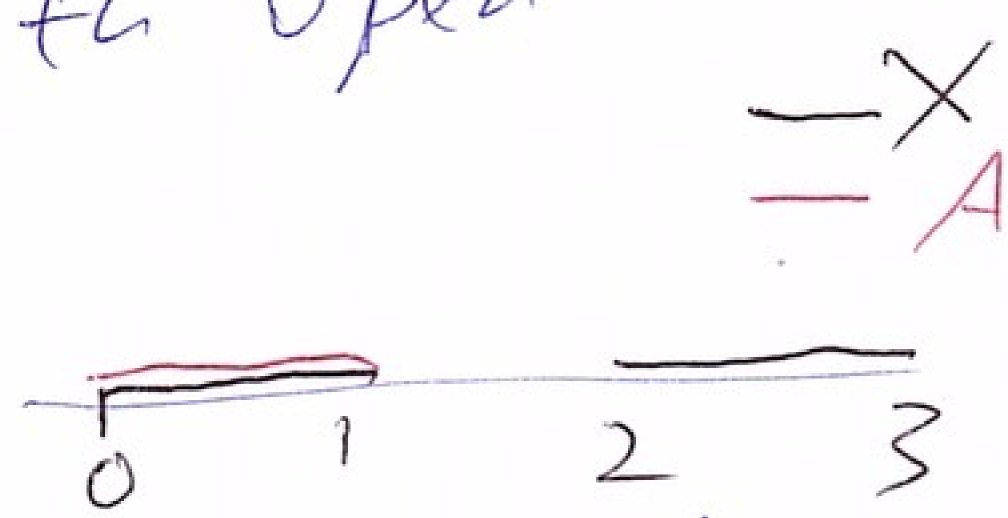
$\square$

Open and closed sets are opposite concepts:

* open sets contain none of their boundaries
* closed sets contain all of their boundaries.

eg: X is both closed and open in (X, d).

The only way A can be both open and closed is if $\partial A = \emptyset$.



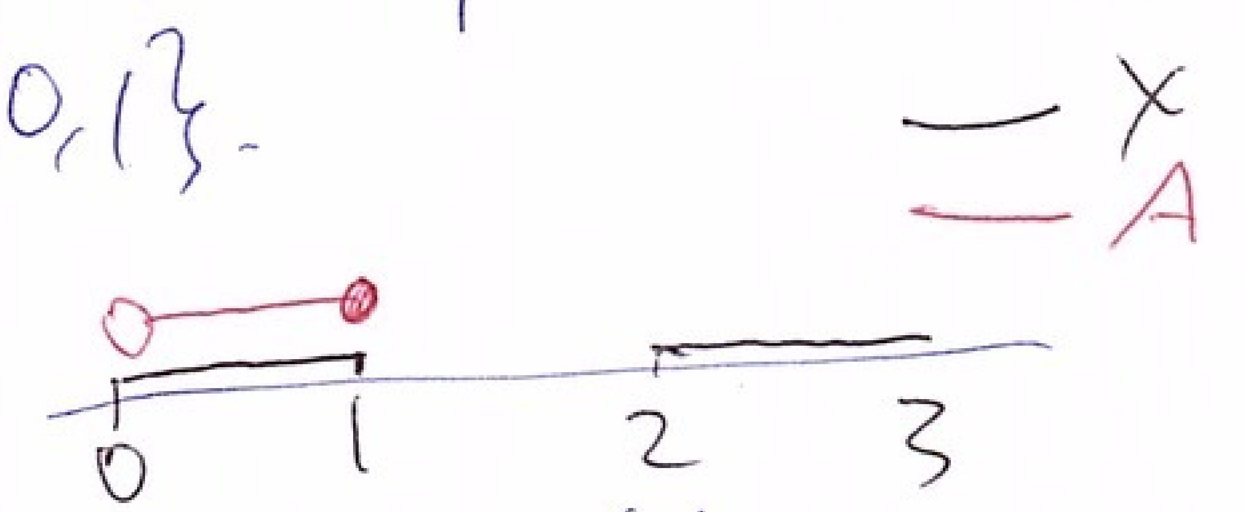
eg: $X = [0,1] \cup [2,3]$.

The set $[0,1]$ is both open and closed in (X, d_2).

If a set contains <u>some but not all</u> of its boundary, then it is neither open nor closed.

eg: $[0,1)$ in $(\mathbb{R}, d_2)$ is neither open nor closed.

e.g. $[0,1]$ is closed but not open $(\mathbb{R}, d_2)$ because $\partial [0,1] = \{0,1\}$.



eg: $X = [0,1] \cup [2,3]$

$\partial A = \{0\}$, $A = (0,1]$. Since $0 \notin A$, A is open but not closed.

Theorem Let A be any set in (X, d). Then A is open iff $X \setminus A$ is closed.

Proof Notice that $\partial A = \partial(X \setminus A)$.

$\Rightarrow$ So if A is open, A contains none of $\partial(X\setminus A)$ $\Rightarrow$ $X \setminus A$ contains all of $\partial(X \setminus A)$.

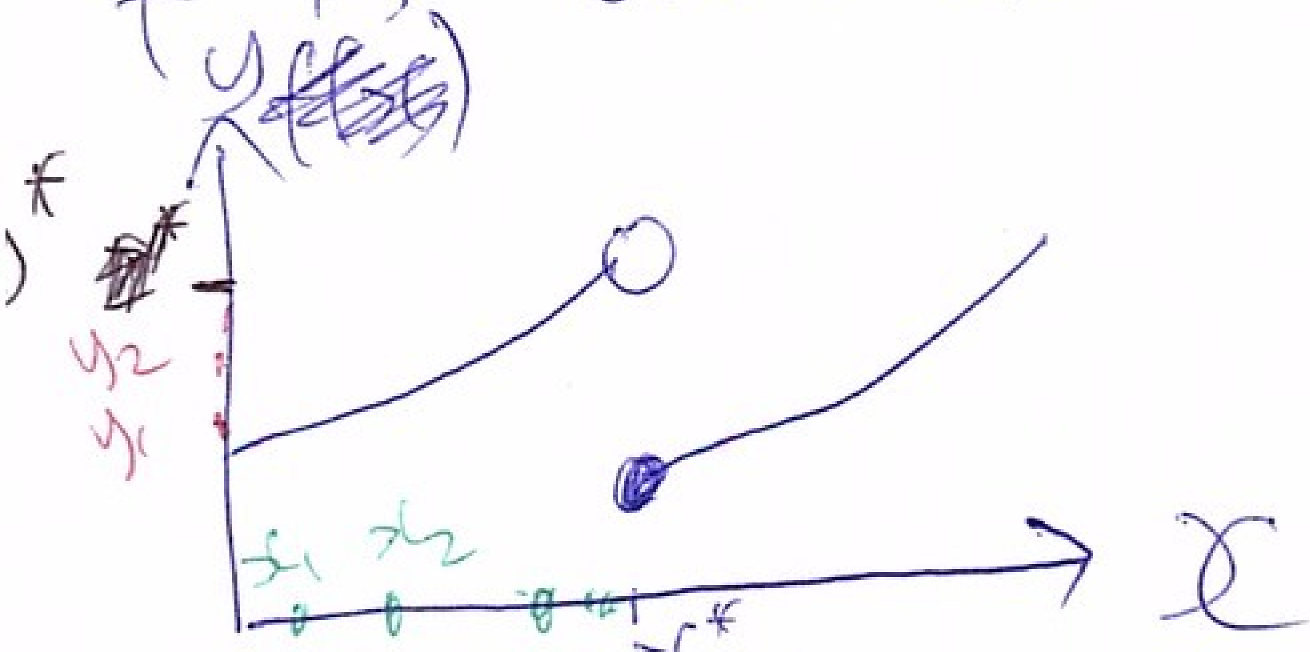
So $X \setminus A$ is closed.

$\Leftarrow$ Similar. $\square$

C.6 Continuity

Def Consider two metric spaces (X, d_X) and (Y, d_Y). We say that $f : X \to Y$ is continuous at $x^* \in X$ if for every convergent sequence $x_n \in X$ with $x_n \to x^*$, the corresponding sequence $y_n = f(x_n) \in Y$ converges to $f(x^*)$.

We say f is continuous if f is continuous for all $x \in X$.



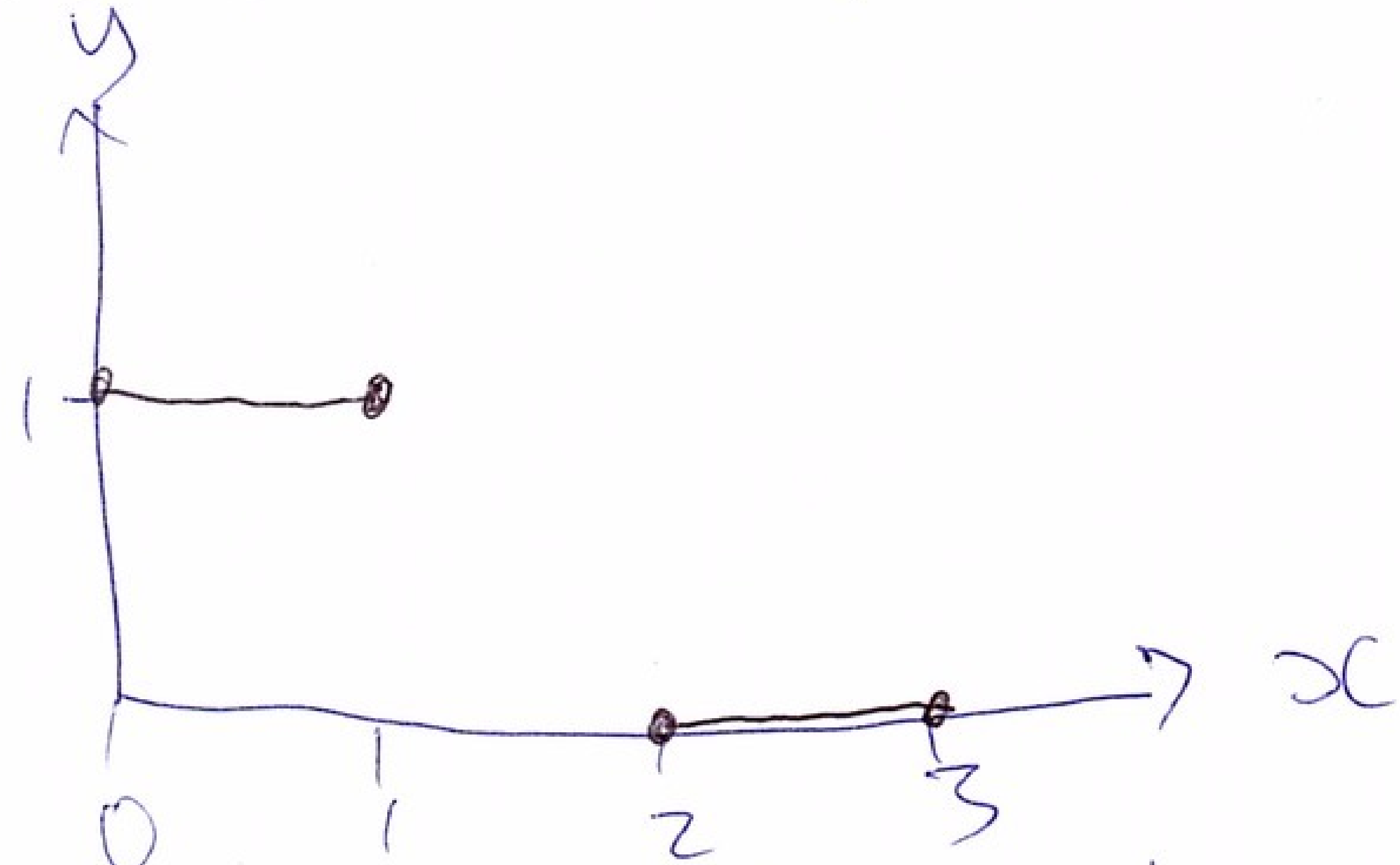
NOT continuous

$x_n \to x^*$

$y_n = f(x_n) \to y^* \neq f(x^*)$.

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \in [2,3] \end{cases}$$

where $X = [0,1] \cup [2,3]$, $Y = \mathbb{R}$.



f is continuous!