C2 Sequences and Convergence

* Cass - Koopmans: $k_n \to k^*$
  capital stock approaches k^*
  No technological progress

* Romer: with progress, k_n grows without bound

* x_n where x_{n+1} is a refined "guess" of a ~~solution~~ guess x_n to solve some problem.

<u>Def</u> A <u>sequence</u> in the set X is any function with domain $\mathbb{N}$ and co-domain X.

Notation: $x_0, x_1, x_2, \ldots$

$\{x_n\}_{n=0}^{\infty}$

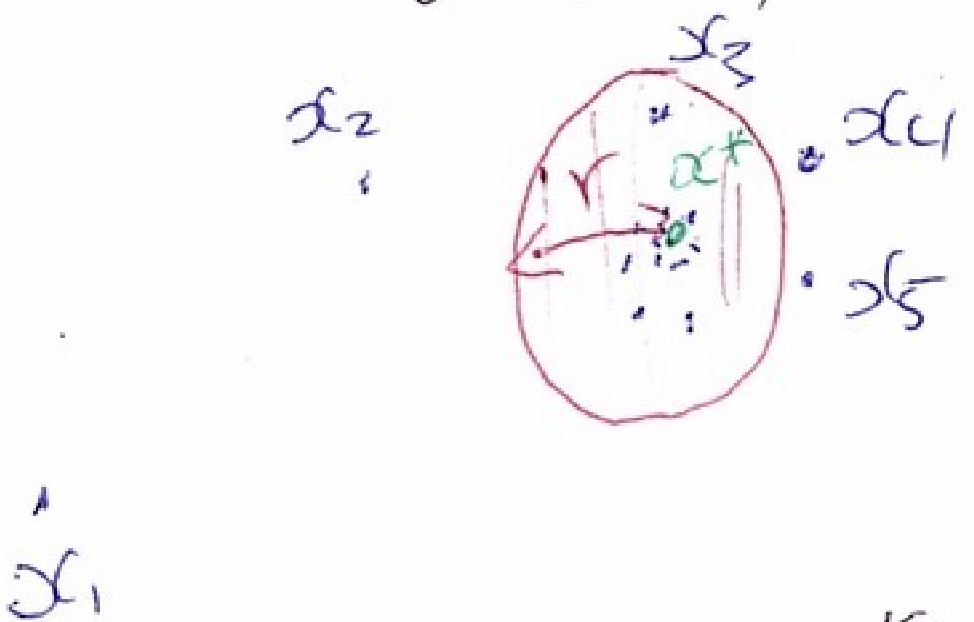
x_n

If x_n and y_n are sequences in $\mathbb{R}$, then you could define $z_n = x_n + y_n$, or $d_n = 2x_n - y_n$.

$x_n = \frac{1}{n}$: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

Def Suppose $x_n \in X$ where (X, d) is a metric space. We say that x_n **converges** to $x^* \in X$ (or write $x_n \to x^*$) if for $r > 0$, there exists a number N such that

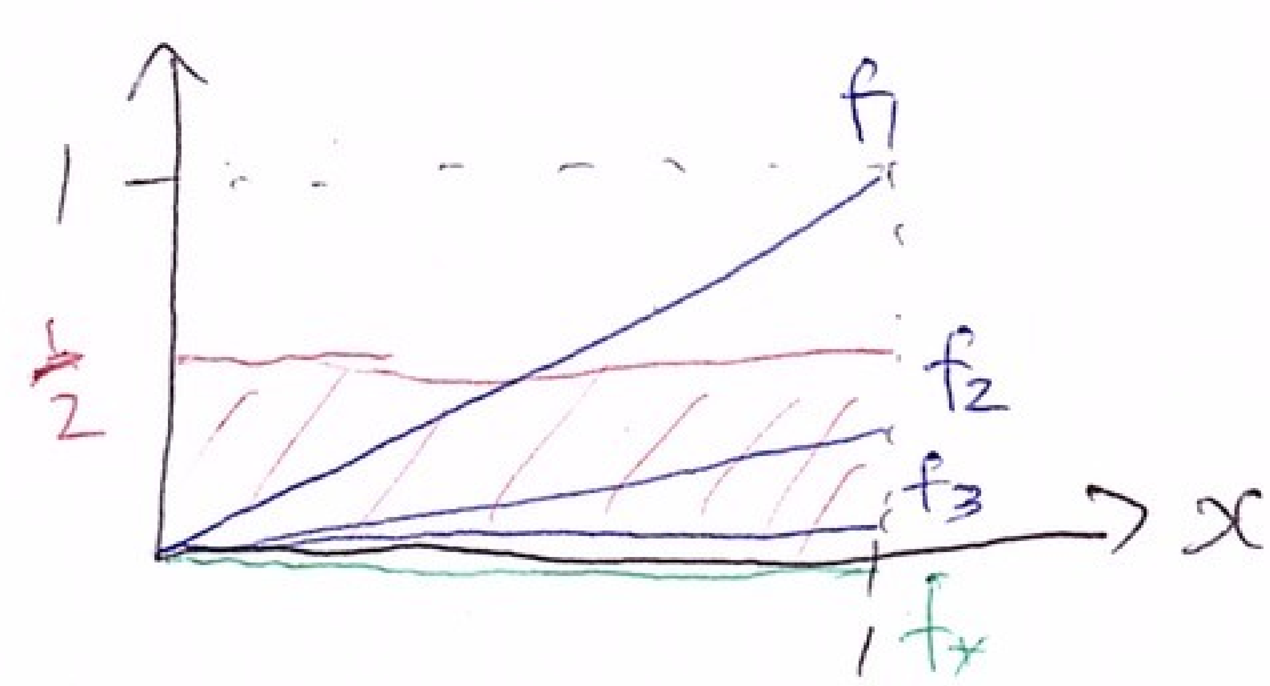
$$d(x_n, x^*) < r \quad \text{for every } n \geq N.$$



In this case, x^* is the limit of x_n. sometimes written $x^* = \lim_{n \to \infty} x_n$.

Examples:

* $x_n = \frac{1}{n}$ in $(\mathbb{R}, d_2)$. $x_n \to 0$.
* $x_n = \frac{1}{n}$ in $(\mathbb{R}_{++}, d_2)$. x_n does not converge.
* $f_n(x) = \frac{x}{n^2}$ in $(\{f : [0,1] \to [0,1]\}, d_\infty)$



$f^*(x) = 0$.

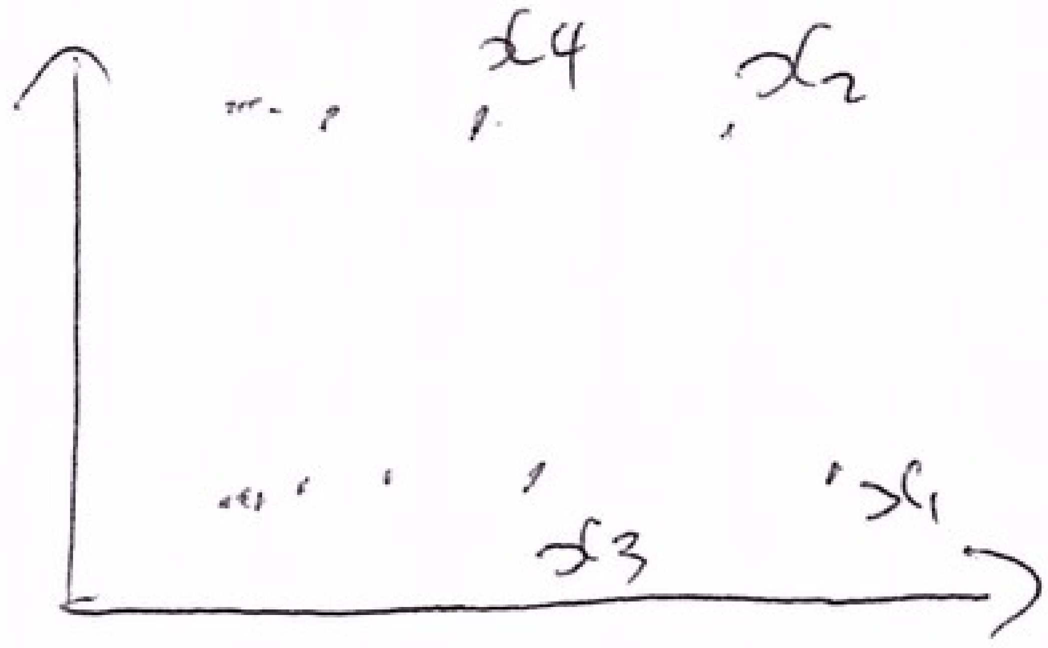
claim: $f_n \to f^*$.

Proof: $d_\infty(f_n, f_m) = d_\infty(f_n(1), f_m(1))$
$= \left| \frac{1}{n^2} - \frac{1}{m^2} \right|$.

Moreover, $d_\infty(f_n, f^*) = \frac{1}{n^2}$.

So given a radius $r > 0$, if we exclude $N = \lceil \sqrt{\frac{1}{r}} \rceil$ items from f_n, then $d_\infty(f_n, f^*) < r$ for $n > N$. □

"ceiling, rounding up."



a non-convergent sequence.

* $x_n = n$ in $(\mathbb{R}, d_2)$ is <u>not</u> convergent.

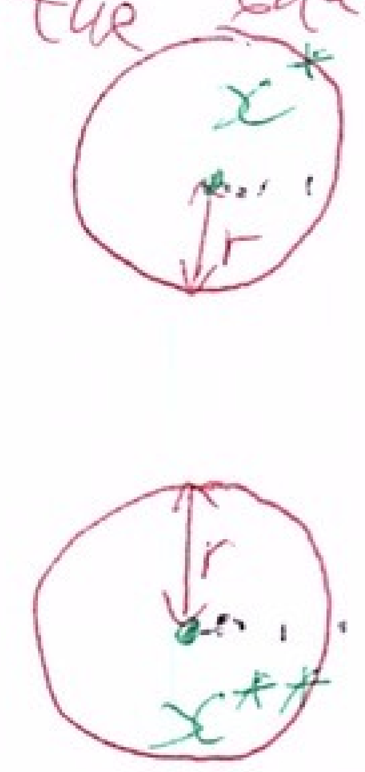
<u>Def</u> Let x_n be a sequence in (X, d). We say x_n is a <u>bounded sequence</u> if there exists some radius $r > 0$ such that $d(x_0, x_n) < r$ for all n. Otherwise, x_n is <u>unbounded</u>.

<u>Theorem</u> If x_n is unbounded, then x_n does not converge.

Contrapositive: If x_n converges, then x_n is bounded.

<u>Theorem</u> A sequence ~~x_n~~ can not converge to more than one point.

"suppose (for the sake of contradiction that"



<u>Proof:</u> Let $r = \frac{1}{2} d(x^*, x^{**})$. Since $\boxed{x_n \to x^* \text{ and } x_n \to x^{**}}$, there is some N such that $d(x_n, x^*) < r$ and $d(x_n, x^{**}) < r$ for all $n \geq N$.

e.g. $r = \frac{1}{3} d(x^*, x^{**})$

We conclude:
$$d(x^*, x^{**}) = r + r > d(x^*, x_N) + d(x_N, x^{**}).$$
This violates the triangle inequality. $\square$

Def We say that y_n is a subsequence of x_n if there exists an increasing sequence $k_n \in \mathbb{N}$ (i.e. $k_{n+1} > k_n$ for all n) such that $y_n = x_{k_n}$.

e.g. $x_n = n^2$, i.e. $0, 1, 4, 9, 16, \dots$

$y_n = 0, 4, 16, \dots$

$y_n = x_{k_n}$ where $k_n = 2n$,

i.e. $k_0 = 0$, $k_1 = 2$, $k_2 = 4$

Theorem If $x_n \to x^*$ and y_n is a subsequence of x_n, then $y_n \to x^*$.

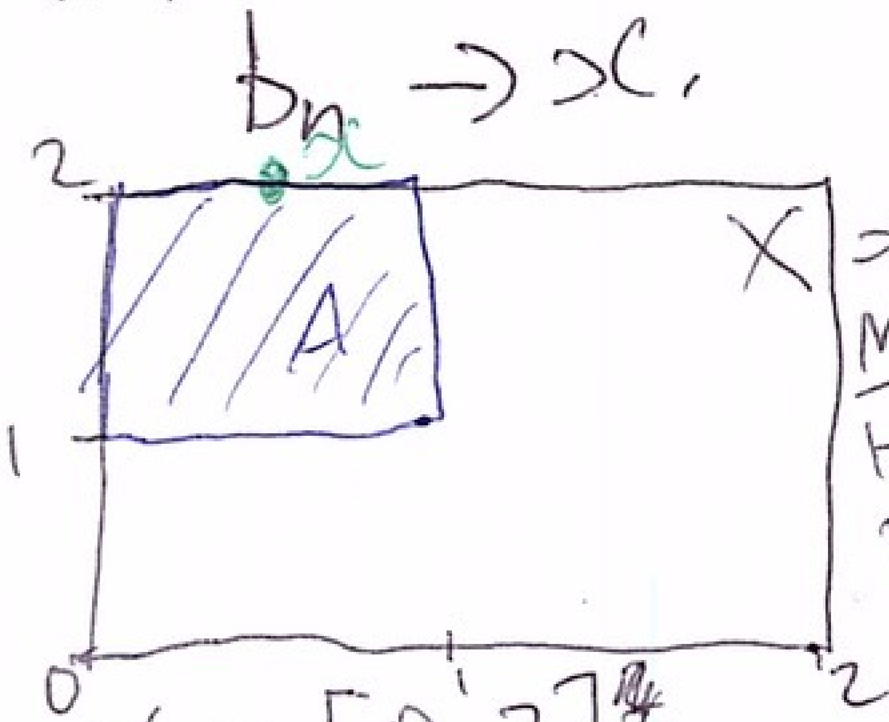
~~Proof~~

C.3 Boundaries

Def Let A be any subset of a metric space (X, d). A point $x \in X$ is a **boundary point** of A if:

(i) there exists a sequence $a_n \in A$ such that $a_n \to x$, AND

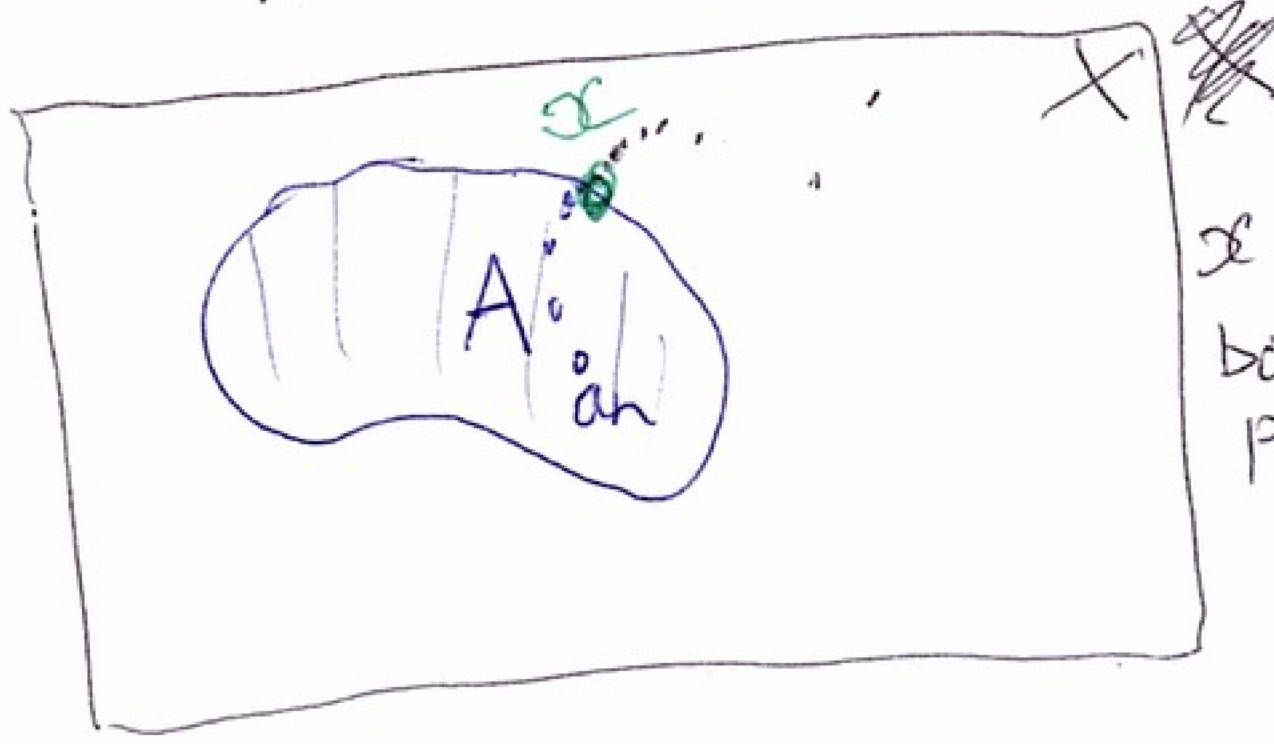
(ii) there exists a sequence $b_n \in X \setminus A$ such that $b_n \to x$.



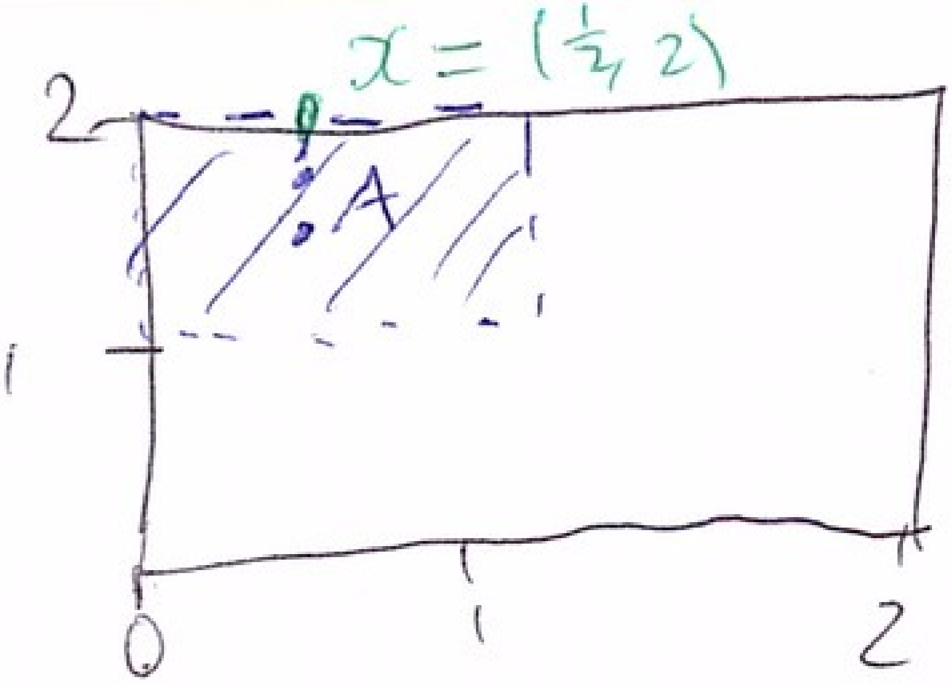
x is NOT a boundary point.

$X = [0, 2]^2$

$A = [0, 1] \times [1, 2]$



x is a boundary point



x is a boundary point:

* e.g. $a_n = (\frac{1}{2}, 2 - \frac{1}{2^{n+2}})$. $a_n \to x$, and $a_n \in A$

* e.g. $b_n = x$.

$X = [0,2] \times [0,2]$

$A = (0,1) \times (1,2)$

where $(0,1) = \{x \in \mathbb{R} : x > 0 \text{ and } x < 1\}$.

<u>Def</u> The set of boundary points of A is called the <u>boundary of A</u>, denoted ∂A.

2.4 Closed sets

<u>Def</u> Suppose A is a subset in a metric space (X, d). We say A is a <u>closed set</u> if there is no sequence $a_n \in A$ such that $a_n \to a^*$ and $a^* \notin A$. "Impossible to escape from A by taking a limit of a sequence in A."

e.g. $[0,1]$ is a closed set in $(\mathbb{R}, d_2)$

X is a closed in (X, d).

$\emptyset$ is a closed in (X, d).

$(0,1)$ is a closed set in $((0,1), d_2)$

but NOT in $(\mathbb{R}, d_2)$.

<u>Theorem</u> Suppose A is a subset of a metric space (X, d). Then A is closed if and only if A contains its boundary, i.e. $\partial A \subseteq A$.

Proof $\Rightarrow$: If A is closed then $\partial A \subseteq A$.

Suppose $x \in \partial A$. By the definition of boundary points, there is a sequence $a_n \in A$ such that $a_n \to x$. Since A is closed $x \in A$. Therefore $\partial A \subseteq A$.

$\Leftarrow$: If $\partial A \subseteq A$, then A is closed.

Suppose $\partial A \subseteq A$ and $a_n \in A$ such that $a_n \to x$. We want to prove $x \in A$. Assume for the sake of contradiction that $x \notin A$. Then the sequence $b_n = x$ satisfies the second critereon of x being a boundary point (i.e. $b_n \notin A$ s.t. $b_n \to x$). ~~Therefore~~ The sequence a_n meets the first critereon. therefore $x \in \partial A \subseteq A$. ↯

Def Let A be a set in (X, d). The **closure** of A is:

$$\mathrm{cl}(A) = \overline{A} = \{x^* \in X : \text{there is a sequence } a_n \in A \text{ such that } a_n \to x^*\}.$$