Chapter 2 Production

2.1 Production functions

N goods
N-1 inputs
1 output

$x \in \mathbb{R}_+^{N-1}$ input
$y = f(x)$ output; production function
$f: \mathbb{R}_+^{N-1} \to \mathbb{R}_+$

Possible assumptions on f:

* possibility of inaction: $f(0) = 0$
$0 = \underbrace{(0, 0, \cdots, 0)}_{N-1 \text{ 0's.}}$

* free disposal (monotonicity):
if $x \geq x'$ (i.e. $x_n \geq x_n'$ for all n)
then $f(x) \geq f(x')$.
* Smoothness: f is twice differentiable.
$\frac{\partial f}{\partial x_i}$ is called the marginal productivity of x_i.
* decreasing marginal productivity:
e.g. $\frac{\partial f(x)}{\partial x_i}$ decreases as x_i increases
and all other ~~the~~ inputs are held constant.
* weakly increasing returns to scale:
For all $x \in \mathbb{R}_+^{N-1}$, and all $t > 1$, $f(tx) \geq t f(x)$.
* constant returns to scale:
For all $x \in \mathbb{R}_+^{N-1}$ and all $t > 1$, $f(tx) = t f(x)$.

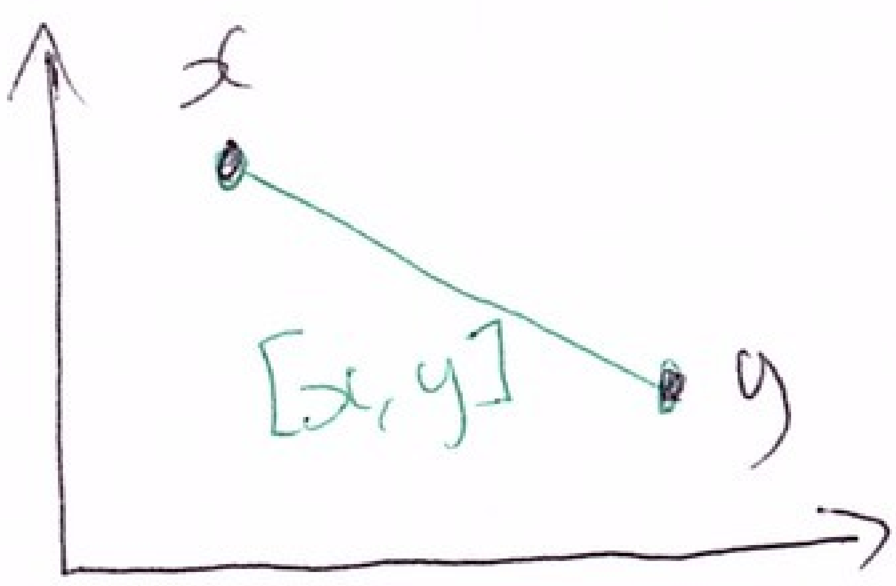
* (weakly) decreasing returns to scale:
For all ~~xxx~~ $x \in \mathbb{R}^{N-1}_+$, and all $t > 1$, $f(tx) \leq t f(x)$

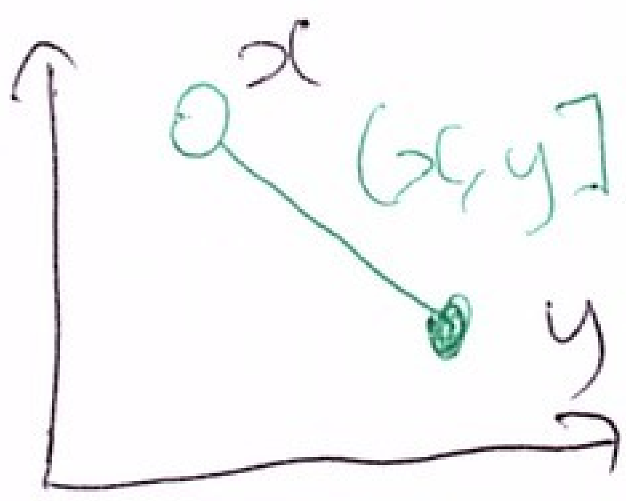
Detour: D convex geometry

Def D.1: A <u>closed interval</u> between $x, y \in \mathbb{R}^n$ is defined as

$$[x,y] = \{\underbrace{ax + (1-a)y}_{\text{convex combination}} : a \in [0,1]\}$$

e.g.: $x = (1,0)$
$y = (0,1)$
$\frac{2}{3}x + \frac{1}{3}y = (\frac{2}{3}, \frac{1}{3})$



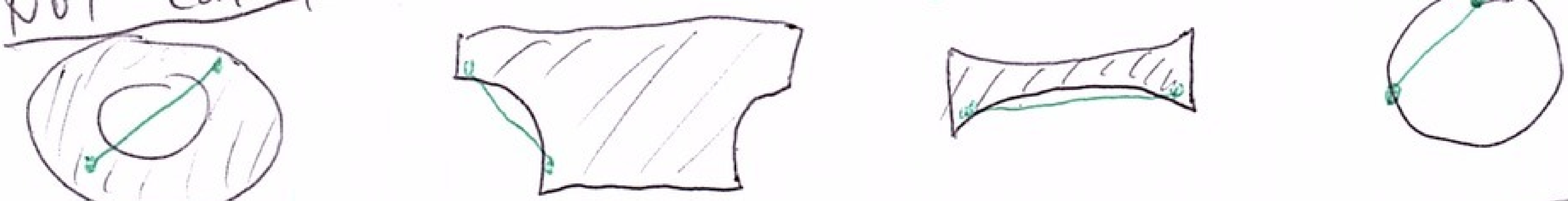


similar definitions for $(x, y]$, etc.

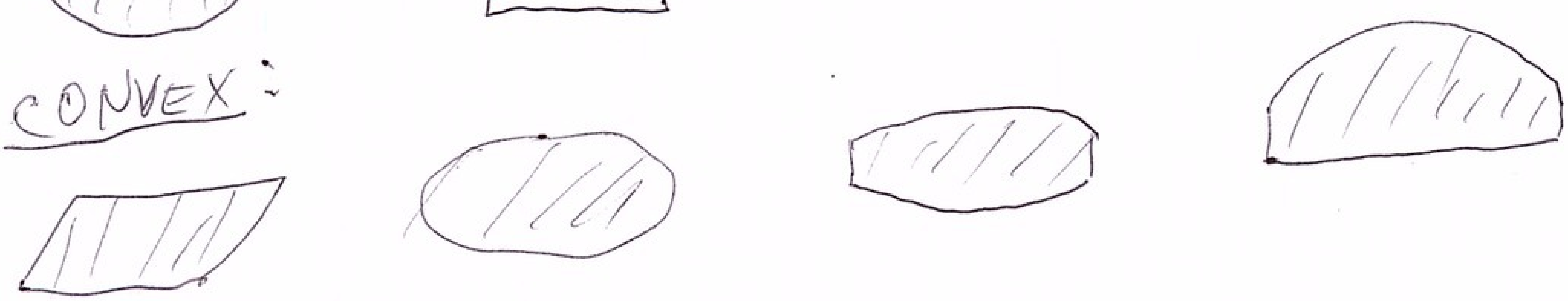
Def D.2: $X \subseteq \mathbb{R}^n$ is a <u>convex set</u> if for all $x, y \in X$, $[x,y] \subseteq X$.

"It is impossible to escape from X by taking convex combinations."
← or drawing a closed interval

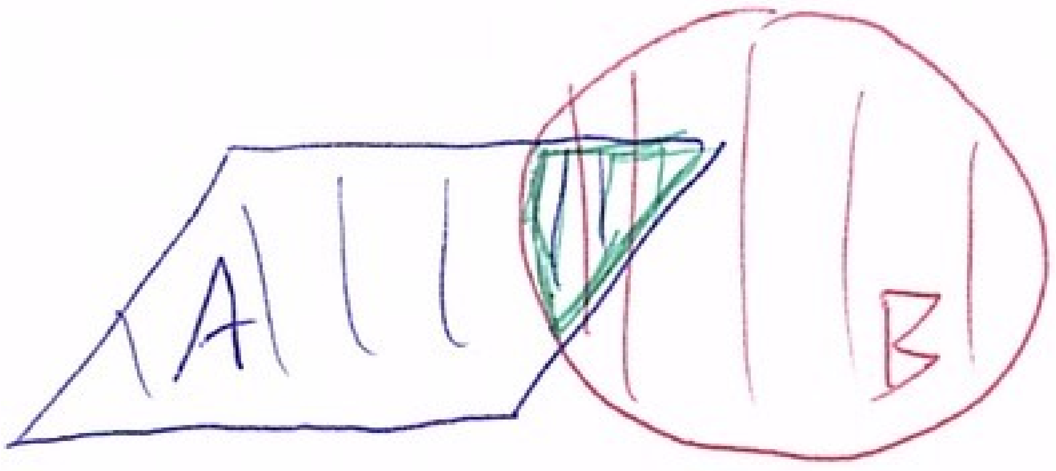
NOT CONVEX:



CONVEX:



Theorem If A and B are convex sets, then $A \cap B$ is a convex set



$\boxed{/\!/}$ $A \cap B$ is convex

Proof Suppose $x, y \in A \cap B$. We want to prove that $[x, y] \subseteq A \cap B$. Since A is convex and $x, y \in A$, we know $[x, y] \subseteq A$. Similarly, $[x, y] \subseteq B$. We conclude $[x, y] \subseteq A \cap B$. $\square$

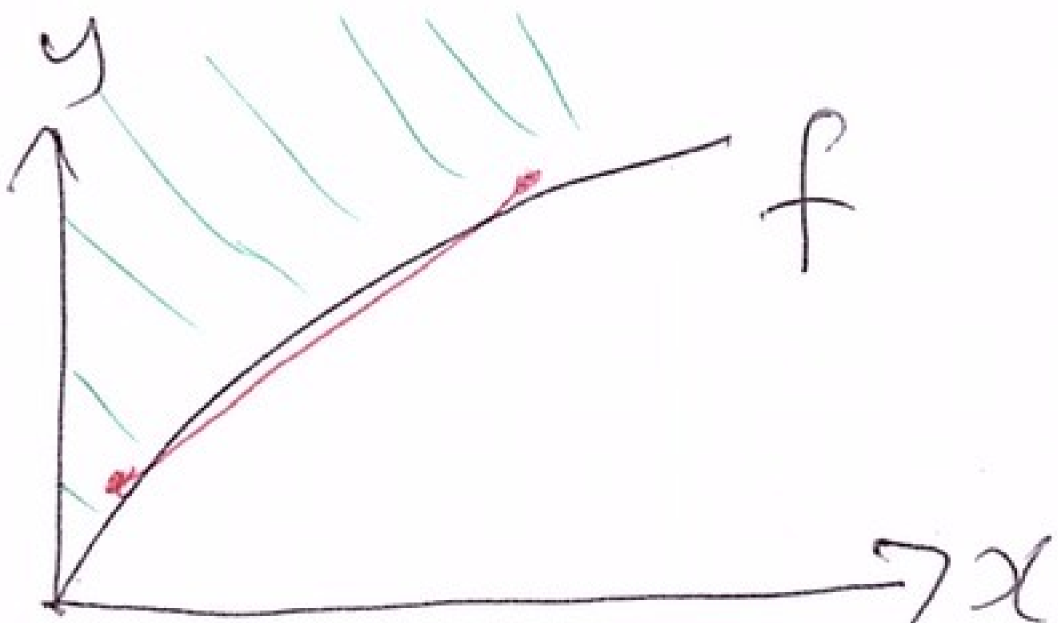
Def $f : X \to \mathbb{R}$ ($X \subseteq \mathbb{R}^n$) is a <u>convex function</u> if its <u>hypergraph</u>,

$$\text{hyper}(f) = \{(x, y) : x \in X,\ y \geq f(x)\}$$

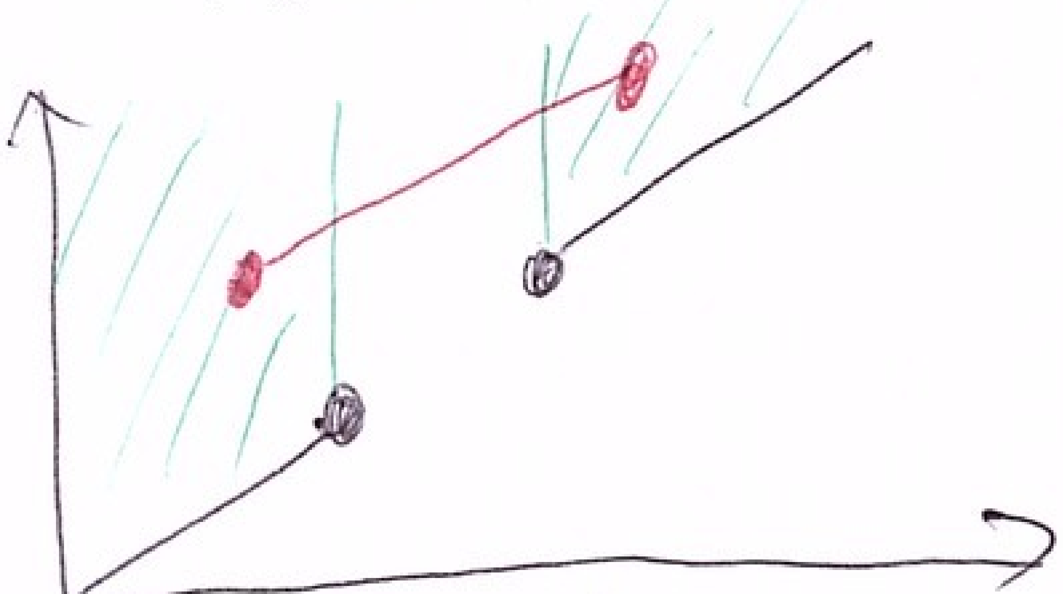
is convex set.

$\boxed{/\!/}$ hyper(f)

f is a convex function



f is NOT a convex function



f is not a convex function

<u>Def</u> $f: X \to \mathbb{R}$ is a <u>concave function</u> if $g(x) = -f(x)$ is a convex function. Or equivalently, if its <u>hypograph</u> is convex, where $hypo(f) = \{(x,y) : y \leq f(x)\}$.

$\boxed{\text{Caution: there is no such thing as a concave set.}}$

<u>Theorem</u> If $f: X \to \mathbb{R}$ is a convex function and $X \subseteq \mathbb{R}^n$ is an open set, then f is continuous.

<u>Theorem</u> Suppose $f: \mathbb{R} \to \mathbb{R}$ is differentiable. Then f is convex if and only if f' is weakly ~~de~~increasing.

<u>Theorem</u> Suppose $f: \mathbb{R} \to \mathbb{R}$ is twice differentiable. Then f is convex if and only if $f''(x) \geq 0$ for all x.

eg: $f(x) = x^2$ is a convex function

$f'(x) = 2x$

$f''(x) = 2$