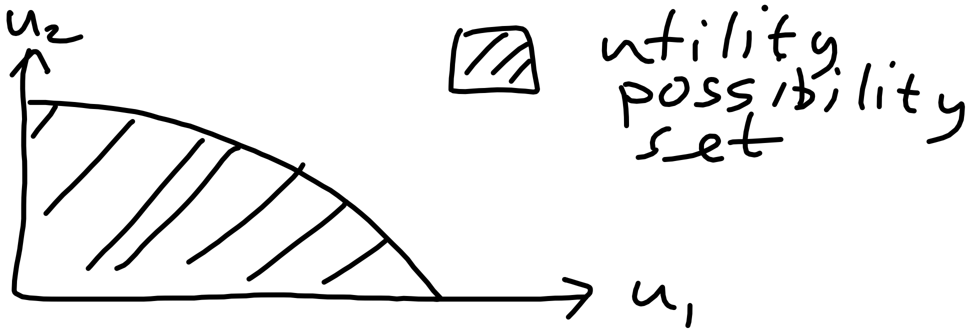
Lecture 2

4.2 Efficient allocations (cont'd.)

last time:



Def. A vector of utilities $u \in \mathbb{R}^{|H|}$ **Pareto dominates** another vector $u' \in \mathbb{R}^H$ if $u > u'$ i.e.

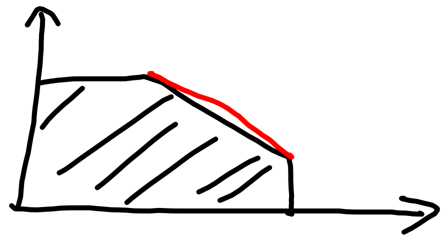
(i) no household is worse off, i.e. $u_h \geq u'_h$ for all $h \in H$,

and

(ii) there is some household h such that $u_h > u'_h$.

If an allocation is Pareto dominated by another feasible allocation, then we say that allocation is <u>inefficient</u>.

If a feasible allocation is not inefficient, then we say it is <u>efficient</u>.



<u>Def</u> The Pareto frontier $\mathcal{U}^*$ is the set of efficient utility vectors.

4.3 Equilibrium

<u>Def</u> Consider a pure exchange economy (u, e). We say (x^*, p^*) consisting of an allocation x^* and a price vector $p^* \in \mathbb{R}_+^N$ is a <u>pure exchange equilibrium</u> if

(i) each household $h \in H$ makes an optimal choice, i.e.

$$x_h^* \in \underset{x_h \in \mathbb{R}_+^N}{\arg\max} \; u_h(x_h)$$
$$\text{s.t. } p^* \cdot x_h \leq p^* \cdot e_h,$$

(ii) all markets clear, i.e.

$$\sum_{h \in H} x_h^* = \sum_{h \in H} e_h$$

$$\Leftrightarrow \sum_{h \in H} x_{hn}^* = \sum_{h \in H} e_{hn} \quad \text{for all } n \in \{1, \dots, N\}$$

4.5 Efficiency of Equilibrium

Theorem Consider a pure exchange economy with increasing utility functions $u_h : \mathbb{R}^N_+ \to \mathbb{R}$ and endowments $e_h \in \mathbb{R}^N_+$. If (x^*, p^*) is an equilibrium, then x^* is an efficient allocation.

Proof Suppose for the sake of contradiction that x^* is inefficient. Then there must be some feasible allocation $\hat{x}$ that dominates x^*.

First, $p^* \cdot \hat{x}_h \geq p^* \cdot x^*_h$ for all $h \in H$.

Second, $p^* \cdot \hat{x}_h > p^* \cdot x^*_h$ for at least one household $h \in H$.

So $\sum_h p^* \cdot \hat{x}_h > \sum_h p^* \cdot x^*_h$.

Since $\hat{x}$ and x^* are feasible, we know

$$\sum_h p^* \cdot e_h = \sum_h p^* \cdot \hat{x}_h = \sum_h p^* \cdot x^*_h$$

Back to B.5

Common form: If A, then $B \Rightarrow C$.

pure exchange economy, increasing utility

equilibrium

efficient

Converse: If A, then $C \Rightarrow B$?

- might not be true.

If the both a statement and its converse are true, we write

If A, then $B \Longleftrightarrow C$.

"if and only if"

iff

"A" is <u>weaker</u> than "B" if $B \Rightarrow A$. (B is <u>stronger</u>)

e.g. a <u>partial converse</u> is a weaker statement than the converse.

eg Consider $x > y \Rightarrow x \neq y$.

Converse: $x \neq y \Rightarrow x > y$ (false)

Partial converse: $x \neq y \Rightarrow x > y$ or $x < y$ (true)

The <u>negation</u> of a statement A is true whenever A is false. We use this in proofs by contradiction. Written $\neg A$.

The <u>contrapositive</u> of "If A then $B \Rightarrow C$" is "If A then $\neg C \Rightarrow \neg B$".

"without loss of generality"
adding an innocuous assumption (that can be dropped)

B.4 Statements & Quantifiers

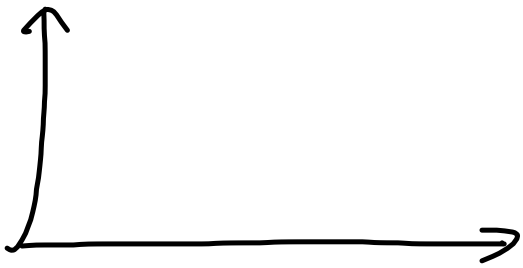
$\cos^2 x + \sin^2 x = 1$
for all $x \in \mathbb{R}$.

← a statement.

Quantifiers: "for all" and "there exists"

$\forall$ $\exists$

sometimes



← 2-dim
space

Metric spaces

e.g. $x, y \in \mathbb{R}^n$

$$d_2(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

by Pythagoras.

Euclidean metric

Define (X, d) is a metric space if X is a non-empty set and the distance metric $d : X \times X \to \mathbb{R}_+$ satisfies:

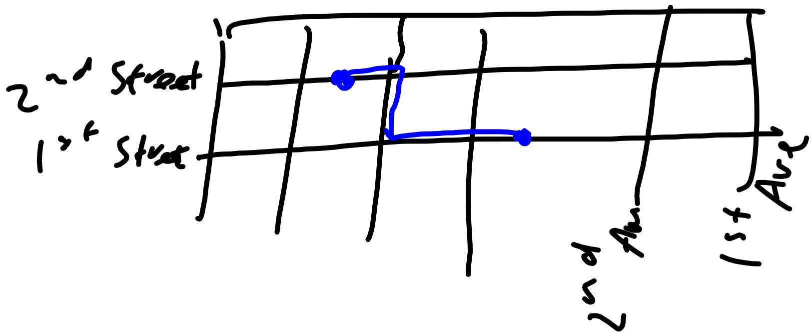
(i) $d(x,y) = 0 \iff x = y$,

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$

(iii) $d(x,z) \leq d(x,y) + d(y,z)$
for all $x,y,z \in X$.
"triangle inequality"
"no short-cuts (y)"

$(\mathbb{R}^n, d_1)$ where

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$



* If (X, d) is a metric space and $Y \subseteq X$ then (Y, d) is a metric space.

abuse of notation

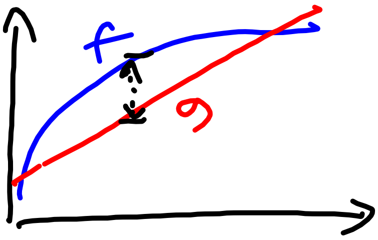
$d_Y : Y \times Y \to \mathbb{R}_+$

* $(\mathbb{R}^n, d_\infty)$ where $d_\infty(x, y) = \max_i |x_i - y_i|$.

* Function spaces.

eg: $X = \{ f : [0,1] \to [0,1] \}$

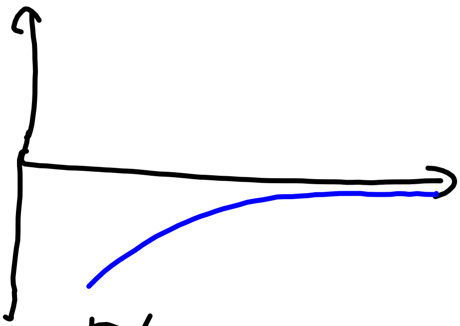
$$d_\infty(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$



like "max", but accommodates no max existing.
"supremum"

supremum vs maximum:

$f(x) = -\frac{1}{x}$, $f: [1,\infty)$



$\max_{x \in [1,\infty)} f(x)$ does not exist.

$\sup = 0$.

Please read C.1.