Proof of Walras' law

(i) Since each household's demand satisfies the budget constraint, we know

$$p \cdot (x_h(p) - e_h) = 0.$$

Summing up:

$$\sum_h p \cdot (x_h(p) - e_h) = 0$$

$$\Leftrightarrow \quad p \cdot \sum_h (x_h(p) - e_h) = 0$$

$$\Leftrightarrow \quad p \cdot z(p) = 0.$$

"harmless assumption"

(ii) Without loss of generality assume that the first N-1 markets clear.
Then $z_j(p) = 0$ for $j \in \{1, \dots, N-1\}$.

Adding up:

$$\sum_{j=1}^{N-1} p_j z_j(p) = 0. \qquad (*)$$

From part (i), we know

$$\sum_{j=1}^{N} p_j z_j(p) = 0. \qquad (**)$$

$(**) - (*)$: $p_N z_N(p) = 0$.

So $z_N(p) = 0$, i.e. N^{th} market clears.

(iii) If there is excess supply or
(trivial) demand in any market at prices p,
then p does not satisfy the
market clearing conditions.

Suppose p is ~~any~~ not an eq. price,
i.e. $z(p) \neq 0 \in \mathbb{R}^N$. We need to
prove $z_j(p) > 0$ and $z_k(p) < 0$ for
some j, k.

For the sake of contradiction,
suppose ~~z_j~~ $z_n(p) \geq 0$ for all n.
(i.e. no excess supply ~~in~~ in any
market).

Then $\sum_{n=1}^{N} p_n z_n(p) > 0$.

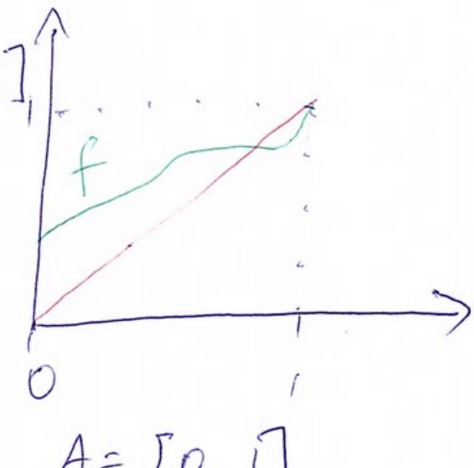
This violates part ~~(i)~~ (i). $\lightning$

Similarly, we rule out $z_n(p) \leq 0$
for all n. $\square$

4.6 Existence of Equilibrium

Brouwer's Fixed Point Theorem

Suppose $A \subseteq \mathbb{R}^n$ is non-empty, convex, and compact and that $f: A \to A$ is continuous. Then f has a fixed point.



Proof for $n=1$; $A=[0,1]$

Let $g(x) = f(x) - x$.
So $g(0) \geq 0$ and $g(1) \leq 0$.
Since g is continuous, by the intermediate value theorem, there is some $x^* \in [0,1]$ such that $g(x^*) = 0$. Therefore, $f(x^*) - x^* = 0$, and hence $x^* = f(x^*)$. So x^* is a fixed point of f. $\square$

Theorem Consider a pure-exchange economy in which each utility function $u_h : \mathbb{R}^N_+ \to \mathbb{R}$ is continuous, strictly increasing, and strictly (quasi-) concave, and that aggregate endowments are strictly positive, i.e. $\sum_h e_{hn} > 0$ for all n. Then there is some equilibrium (p^*, x^*).

<u>Proof</u> The assumptions on u_n are purely to establish that excess demand $z: \mathbb{R}^N_{++} \to \mathbb{R}^N$ is continuous.

Recall p^* is an equilibrium price vector iff $z(p^*) = 0$. $\leftarrow \in \mathbb{R}^N$

Define truncated excess demand:

$$\bar{z}_i(p) = \min\{1, z_i(p)\}.$$

Note: $\bar{z}$ is still continuous and p^* is an eq. price $\Leftrightarrow \bar{z}(p^*) = 0$.

Define: $$p_i' = p_i + \underbrace{\max\{0, \bar{z}_i(p)\}}_{\geq 0}$$

By Walras law, ~~if~~ $p = p'$ ~~then~~ $\Leftrightarrow$ ~~p is an~~ $z(p) = 0 \Leftrightarrow p$ is an eq. price.

Define: $$p_i'' = \frac{p_i'}{\sum_{j=1}^N p_j'}. \leftarrow \sum_{i=1}^N p_i'' = 1.$$

We have a continuous function

$$f: \underbrace{[0,1]^N}_{\text{old price}} \to \underbrace{[0,1]^N}_{\text{new price}}.$$

By Brouwer, f has a fixed point p^*. So $\bar{z}(p^*) = 0$ and $z(p^*) = 0$ ~~as~~ so $(p^*, x(p^*))$ is an equilibrium. $\square$

~~4.7~~ 4.7 Second welfare theorem

Def Pure exchange equilibrium with lump-sum taxes

Consider a pure exchange economy with N goods, utility functions $\{u_h\}_{h \in H}$ and endowments $\{e_h\}_{h \in H}$.
A feasible allocation $(\{x_h^*\}_{h \in H}, p^*)$ is a pure exchange equilibrium with lump-sum taxes $\{t_h\}_{h \in H}$ if

(i) $\sum_{h \in H} t_h = 0$

(ii) $x_h^* \in \arg\max_{x_h \in \mathbb{R}^N_+} u_h(x_h)$
s.t. $p^* \cdot x_h^* = p^* \cdot e_h - t_h$

(iii) markets clear.

Theorem Consider a pure-exchange economy with continuous, increasing, strictly concave utility functions and strictly positive agg. endowments (same as existence theorem).
If $x^* \in \mathbb{R}^{NH}_{++}$ is an efficient allocation, ~~and~~ then there exists prices p^* and

taxes t^* s.t. (x^*, p^*, t^*) is a pure-exchange eq. w/ lump sum taxes.

Proof

Consider the economy with same utility functions, but endowments $= x^*$.
By existence theorem, there are prices p^* and allocation x^{**} s.t. (p^*, x^{**}) is a pure-exchange equilibrium.
No-one can be worse off under x^{**} (because households can refuse to trade). Since x^*, no-one x^{**} can be better off either. So everyone is indifferent between x^* and x^{**}.
We conclude that (p^*, x^*) is an eq.

Let $t_h^* = p^* \cdot e_h - p^* \cdot x_h^*$.
So the budget constraints with taxes t^* are the same (for each household) as when endowments $= x^*$.
So (p^*, x^*, t^*) is an eq w/ lump-sum taxes.

$\square$