<u>Q.11</u> Let (X, d) be a discrete metric space. Let $A \subseteq X$. What is ∂A?

<u>Answer</u>: $\partial A = \emptyset$.

<u>Proof</u> Suppose $x \in \partial A$.

That means:

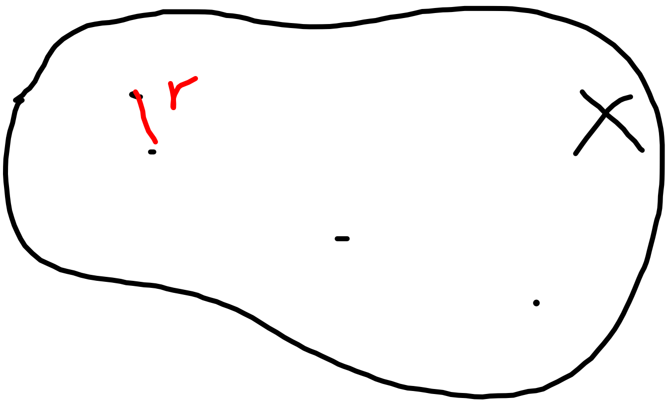
* $a_n \in A$ with $a_n \to x$.
  $\hat{a}_n = x$ is a subsequence.
  $\Rightarrow x \in A$.
* $b_n \in X \setminus A$ with $b_n \to x$.
  $\hat{b}_n = x$ is a subsequence.
  $\Rightarrow x \notin A$.

Since $x \in \partial A$ requires

$x \in A$ and $x \notin A$, we conclude $\partial A = \emptyset$. $\square$

A.15 Let (X, d) be a metric space. Prove: If $A \subseteq X$ is a finite set, then A is closed.

Proof



Let $r = \min\limits_{x,y \in A} d(x,y)$.

Let $x_n \in A$ be a convergent sequence. Thus, x_n is a Cauchy sequence. For some N,

$$d(x_n, x_m) < r \quad \text{for all } n,m > N$$

" " "

$$\Rightarrow x_n = x_m$$

$$\Rightarrow x_n = x^* \quad \text{for some } x^* \in A \text{ and all } n > N$$

$$\Rightarrow \lim_{n \to \infty} x_n \in A.$$

So A is closed. $\square$

Another answer:

$\{a\}$ is a closed set for all $a \in A$.

Since A is finite,

$$\bigcup_{a \in A} \{a\} = A$$

is closed. $\square$

A.27 Counter-example to:
Let (X, d) be a metric space. If $A \subseteq X$ is open, then

$$\operatorname{int}(\operatorname{cl}(A)) = A.$$

Counterexample: $A = (0,2) \setminus \{1\}$.
This is open, since $(0,1)$ and $(1,2)$ are open balls, and A is the union of these two open sets.

$$\operatorname{cl}(A) = [0,2].$$

$$\operatorname{int}(\operatorname{cl}(A)) = (0,2) \neq A. \quad \square$$

A.58 Let (X, d) be a metric space. Consider two sequences $x_n, y_n \in X$.

Prove: If $d(x_n, y_n) \xrightarrow[d_2]{} 0$ and $x_n \to x^*$, then $y_n \to x^*$.

Proof Pick any radius r.
We want to find N s.t.
$d(y_n, x^*) < r$ for all $n > N$.

① Pick N_1 s.t. $d(x_n, y_n) < \frac{r}{2}$ for $n > N_1$.

② Pick N_2 s.t. $d(x_n, x^*) < \frac{r}{2}$ for $n > N_2$.

Pick $N = \max\{N_1, N_2\}$.

This means for all $n > N$,

$$\begin{aligned} d(y_n, x^*) &\leq d(y_n, x_n) + d(x_n, x^*) \quad \text{triangle inequality} \\ &< \frac{r}{2} \text{ ①} + \frac{r}{2} \text{ ②} \\ &= r. \end{aligned}$$

Thus $y_n \to x^*$. $\square$

2.1 If $f: \mathbb{R}_+^{N-1} \to \mathbb{R}$ is smooth and f has constant returns to scale, then

ambiguous
$$\frac{\partial f(tx)}{\partial x_i} = \frac{\partial f(x)}{\partial x_i}$$

for all $t>0$ and all $x \in \mathbb{R}_+^{N-1}$.

Proof CRTS means $f(tx) = t f(x)$.

$$\Rightarrow \frac{\partial}{\partial x_i}\left[f(tx)\right] = \frac{\partial}{\partial x_i}\left[t f(x)\right]$$

$$f_i(tx) = f_i(x)$$

$$\left[\frac{\partial f(x')}{\partial x_i'}\right]_{x'=tx} = \frac{\partial f(x)}{\partial x_i}.$$

$$\cancel{t} f_i(tx) = \cancel{t} f_i(x)$$

$$f_i(tx) = f_i(x).$$

<u>2.2</u> Find a production function with decreasing marginal productivity and increasing returns to scale.

<u>Proof</u>: $f(x, y) = (xy)^{\frac{2}{3}}$

$\downarrow$ MP: $\dfrac{\partial f(x,y)}{\partial x} = \dfrac{2}{3} x^{-\frac{1}{3}} y^{\frac{2}{3}}$

$$= \frac{2}{3} \underbrace{\frac{y^{\frac{2}{3}}}{x^{\frac{1}{3}}}}_{\text{decreasing in } x.}$$

$\uparrow$ RTS: $f(tx, ty)$

$$= (tx \cdot ty)^{2/3}$$

$$= [(t^2)\, xy]^{2/3}$$

$$= t^{4/3}\, (xy)^{2/3}$$

$$> (xy)^{2/3} \quad \text{for } t > 1$$

$$= f(x, y). \qquad \square$$

2.7 Corner shop.

w wages

ℓ Labour

p_x^g price of good $g \in \{m, c\}$
in market $x \in \{W, R\}$

q_x^g quantity of ".

$f(q^c, q^m, \ell)$ choc sold

$h(q^c, q^m, \ell)$ milk sold

Profit:

$$\pi(w, p_W^c, p_R^c, p_W^m, p_R^m)$$

$$= \max_{l, q^c, q^m} p_R^c f(q^c, q^m, l) + p_R^m h(q^c, q^m, l) - wl - p_W^c q^c - p_W^m q^m.$$

If $p_W^g > p_R^g$ for both $g \in \{c, m\}$, then the firm would shutdown, i.e. $q^c = q^m = 0$.

FOC's don't apply when boundary choices are optimal.