# A.43 $(X, d)$ compact $\Rightarrow$ complete

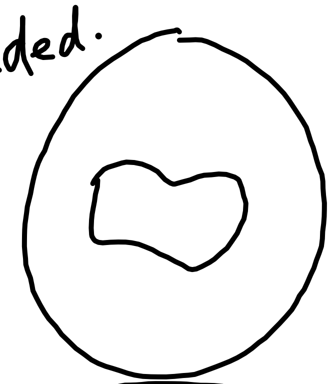
Suppose $x_n \in X$ is a Cauchy sequence. We want to show that $x_n$ is convergent.

Since $(X,d)$ is compact, $x_n$ has a convergent subsequence, $y_n \to y^* \in X$.

Since $x_n$ is Cauchy, Theorem A.13 implies that $x_n \to y^*$, i.e. $x_n$ is convergent. So $(X, d)$ is compact. $\square$

A.44 In $(\mathbb{R}, d_2)$, compact?

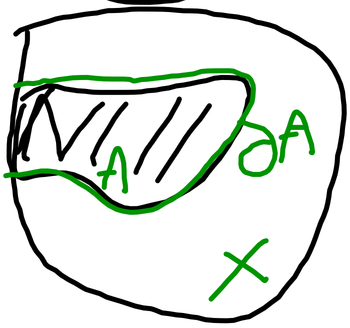
$\emptyset$ ✓ closed & bounded.



bounded set (✓ X?)

boundary of a set

confusion




$\mathbb{R}$ X unbounded

$\{0\}$ only seq in this set is $x_n = 0$. Now $x_n \to 0 \in \{0\}$.

Every seq. is convergent.

$[0,1)$ ✗ not closed

$x_n = 1 - \frac{1}{n} \to 1 \notin [0,1)$.

$[0,1]$ ✓ closed & bounded.

$\underline{\text{Ex1}\ \ (\mathbb{R}_{++}, d_2)}$ not Euclidean!

$(0,1)$ ✗ not compact.

$x_n = \frac{1}{n}$ does not have a convergent subsequence.

$(0,1]$ ✗ closed, bounded, NOT compact.

Is closed:

$\partial(0,1] = \{1\}. \subseteq (0,1]$ ✓

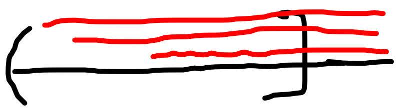
Is bounded:

$(0,1] \subseteq N_1(1) = (0,2).$

Not compact:

opt1 $x_n = \frac{1}{n}$ has no convergent subsequence.

opt2 $\left\{ N_{1-\frac{1}{n+1}}(1) \right\}$



has no finite subcover.

A.45 $(X,d)$ compact, $K$ is a closed set, then $K$ is compact.

Let $x_n \in K$. I want to show $x_n$ has a convergent subsequence, with limit in $K$.

Since $x_n \in X$, and $X$ is compact $x_n$ has a subsequence $y_n \to y^* \in X$.

Want to prove $y^* \in K$.

Since $y_n \in K$ and $K$ closed, $\lim y_n \in K$. $\square$

# Answer 2

Want to prove: every open cover $C$ has a finite subcover.

Let $C' = C \cup \{X \setminus K\}$, where $X \setminus K$ is open since $K$ is closed.

$C'$ is an open cover of $X$.

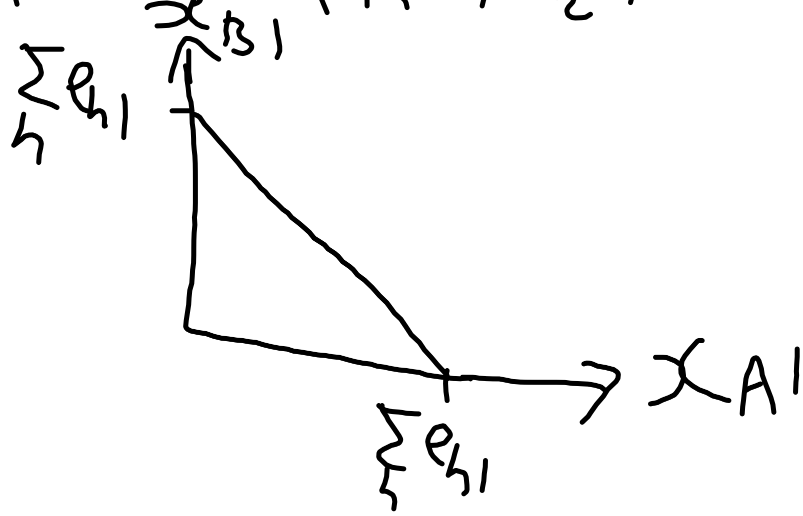
Since $X$ is compact, it has a finite subcover $\{U_1, \ldots, U_N\} = C''$

If $X \setminus K \in \mathcal{C}''$, we can throw it out!
So what's left is an open subcover of $K$. $\square$

A.49 In a pure-exchange e-n with $N$ goods, $H$ households, show that the set of feasible allocations, $A$, is compact inside $(\mathbb{R}_+^{HN}, d_2)$.

To use B-W, we need to check

$A$ is closed and bounded inside $(\mathbb{R}^{HN}, d_2)$.



$$f: \underset{\text{allocations}}{\mathbb{R}^{HN}} \to \underset{\text{goods}}{\mathbb{R}^{N}}$$

$$f((x_{hn})) = \sum_h (e_h - x_h)$$

vectors

$(x_{hn})$ is feasible, i.e. $\in A$,

$$\Leftrightarrow f((x_{hn})) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\text{So } A = f^{-1}\left(\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}\right\}\right)$$

closed set

Since $f$ is continuous,
$f^{-1}$(closed set) is closed.

$\Rightarrow A$ is closed.