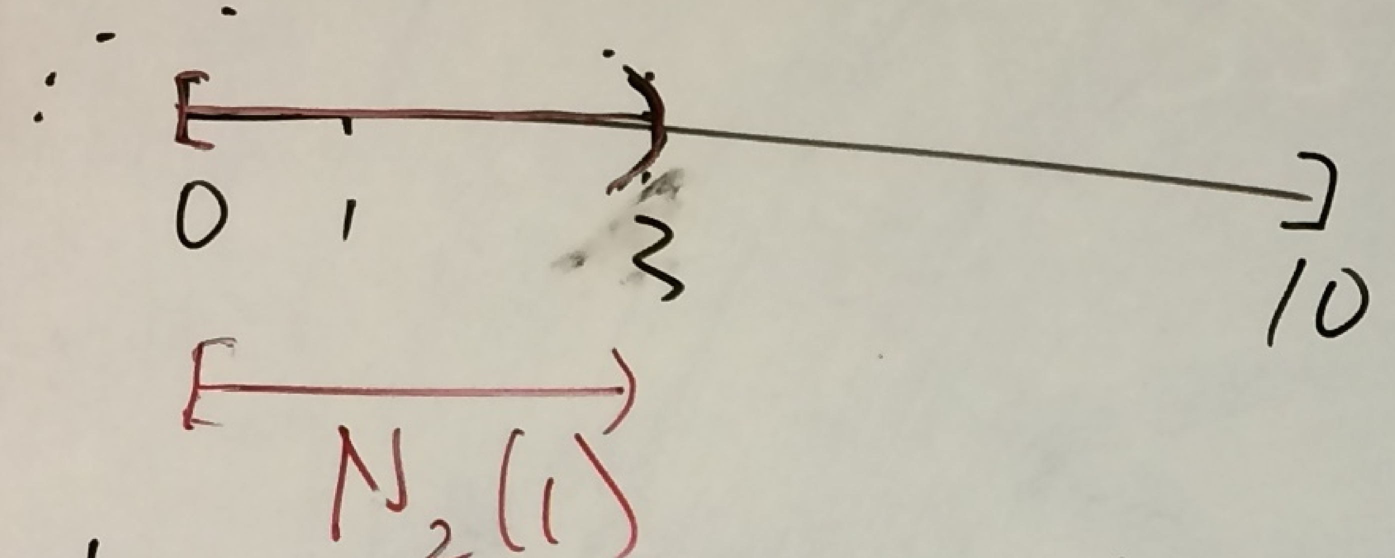
$\underline{A.20}$ Let $(X, d) = ([0,10], d_2)$.

(i) $A = N_2(1) = [0, 3)$.

radius $(\in \mathbb{R})$ — centre $\in X$



(ii) Is this set open?

A: Yes. Open balls are open.

(iii) Is A open in $(\mathbb{R}, d_2)$?

A: No. $\partial A = \{0, 3\}$, so $A \cap \partial A = \{0\}$, so A contains some of its boundary.

$\Rightarrow$ A is not an open set.

Another approach: $N_r(0) \nsubseteq A$ for any radius r.

$\underline{A.23}$ If A, B are open sets in (X, d), then $A \cap B$ is also open.

$\underline{\text{Proof:}}$

For any $x \in A \cap B$, we would like to find a radius r s.t. $N_r(x) \subseteq A \cap B$.

Since A is open, there is a radius r_A s.t. $N_{r_A}(x) \subseteq A$.

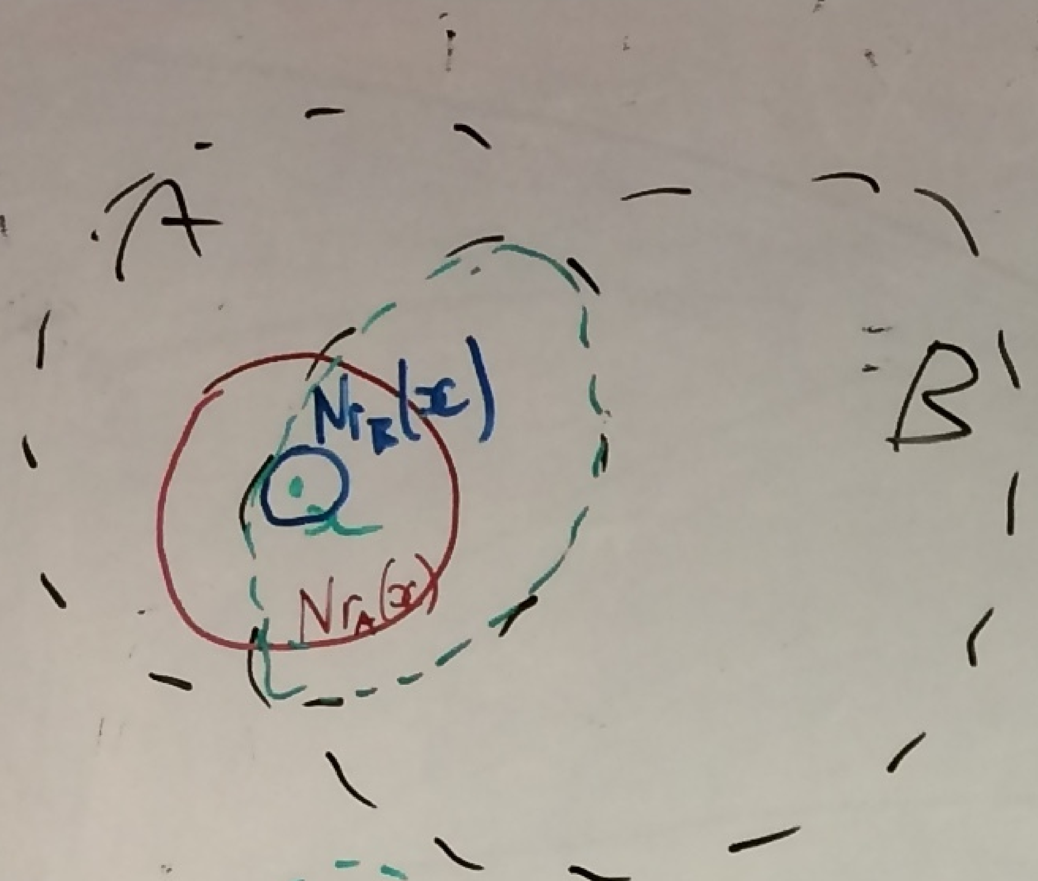
There's a similar radius r_B for B.

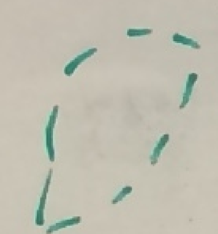
Let $r = \min\{r_A, r_B\}$.

Notice that $N_r(x) \subseteq N_{r_A}(x)$ and $N_r(x) \subseteq N_{r_B}(x)$.

We conclude $N_r(x) \subseteq A \cap B$.

Therefore $A \cap B$ is open. $\square$



$O \subseteq$ 

A.24 False conjecture. Let $\mathcal{A}$ be a set of open sets in (X, d). Then $\bigcap_{A \in \mathcal{A}} A$ is open.

Counter ~~example~~: $(X, d) = (\mathbb{R}, d_2)$.

$(-1, 1) \cap (-\frac{1}{2}, \frac{1}{2}) \cap (-\frac{1}{3}, \frac{1}{3}) \cap \ldots$

$= \bigcap \{(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$

$= \{0\}$.

Not open.

A.28 Suppose $u : \mathbb{R}^N_+ \to \mathbb{R}$ is continuous.

(i) Prove that each indifference curve

$$I(\bar{u}) = \{x \in \mathbb{R}^N_+ : u(x) = \bar{u}\}$$

target utility

is a closed set in $(\mathbb{R}^N_+, d_2)$.

Proof: $\{\bar{u}\}$ is a closed set in $(\mathbb{R}, d)$. Since u is continuous, $u^{-1}(\{\bar{u}\})$ is a closed set.

Notice $u^{-1}(\{\bar{u}\}) = \{x \in \mathbb{R}^N_+ : u(x) \in \{\bar{u}\}\}$

$= I(\bar{u})$. $\square$

(ii) Prove that each upper contour set $V(\bar{u}) = \{x \in \mathbb{R}^n_+ : u(x) \geq \bar{u}\}$ is a closed set.

Proof $[\bar{u}, \infty)$ is a closed set. Since u is continuous, $u^{-1}([\bar{u}, \infty))$ is closed. Since $u^{-1}([\bar{u}, \infty)) = V(\bar{u})$, $V(\bar{u})$ is closed. □

A.29 Suppose (X, d_X), (Y, d_Y), (Z, d_Z) are metric spaces and $f: X \to Y$ and $g: Y \to Z$ are continuous. Prove that

$$h: X \to Z$$
$$h(x) = g(f(x))$$

is continuous.

Proof Let $x_n \to x^*$ be a seqence in (X, d).
We want to prove that $h(x_n) \to h(x^*)$.
Since f is continuous, $f(x_n) \to f(x^*)$.
Since g is continuous, $g(f(x_n)) \to g(f(x^*))$.

The

A

Therefore, $h(x_n) \to h(x)$. $\square$

<u>A.20</u> $(X, d) = ([0,10], d_2)$

$A = [0,3)$.

$\partial A = \{3\}$.

Why is $0 \notin \partial A$?

For 0 to be a boundary point, need:

(i) $a_n \in A$ s.t. $a_n \to 0$. ✓

$a_n = \frac{1}{n}$.

(ii) $b_n \in X \setminus A = [3,10]$ s.t. $b_n \to 0$. ✗

<u>A.30</u> Let (X,d) be a metric space. Fix any $x_0 \in X$, let $f: X \to \mathbb{R}$ defined by $f(x) = d(x, x_0)$.
Prove that f is continuous.

<u>Proof</u> Let $a_n \to a^*$ be a sequence in (X,d).
Want to show $f(a_n) \to f(a^*)$.
By the triangle inequality,

$$f(a_n) = d(a_n, x_0) \leq d(a_n, a^*) + d(a^*, x_0)$$
$$= d(a_n, a^*) + f(a^*).$$

Similarly,

$$f(a^*) = d(a^*, x_0) \leq d(a^*, a_n) + d(a_n, x_0)$$
$$= d(a_n, a^*) + f(a_n).$$

Rearranging,

$$f(a_n) - f(a^*) \leq d(a_n, a^*)$$
$$f(a^*) - f(a_n) \leq d(a_n, a^*)$$
$$\Rightarrow |f(a_n) - f(a^*)| \leq d(a_n, a^*).$$

Since $a_n \to a^*$, for all $r > 0$, there is an N s.t.

$$d(a_n, a^*) < r \quad \text{for all } n > N$$
$$\Rightarrow |f(a_n) - f(a^*)| < r \quad \text{" "}$$
$$\Rightarrow f(a_n) \to f(a^*).$$

$\square$