Q.1 Prove that the cake-eating value $V(k)$ is strictly increasing.

Proof Suppose $k_1 < k_2$.

Given any v, want to prove

$$F(v)(k_1) < F(v)(k_2)$$

where F is the Bellman op.

$$F(v)(k) = \max_{x,\, k'} u(x) + \beta v(k') \quad \text{s.t. } x + k' = k.$$

If this property is true,
then $V^* = F(V^*)$
$\Rightarrow V^*$ is strictly increasing.

$$F(V)(k_2) = \max_{x,k'} u(x) + \beta V(k') \quad \text{s.t. } x + k' = k_2$$

$$\geq \max_{x,k'} u(x + [k_2 - k_1]) + \beta V(k') \quad \text{s.t. } x + k' = k_1$$

$$> \max_{x,k'} u(x) + \beta V(k') \quad \text{s.t. } x + k' = k_1$$

$$= F(v)(k_1). \qquad \square$$

Alternate answer:

Let $X = \{ f \in CB(\mathbb{R}_+) : f \text{ is weakly increasing} \}$.

<u>Claim</u>: (X, d_∞) is a complete metric space.

Strategy: if we can prove

$F : X \to X$

weakly value functions

Q.4 Suppose cake grows by $1+r$ every day.
Write down the Bellman eq.

$$V(k) = \max_{x, k' \geq 0} u(x) + \beta V(k'(1+r)) \quad \text{s.t. } x + k' = k$$

$$= \max_{x, k' \geq 0} u(x) + \beta V(k') \quad \text{s.t. } x + \frac{k'}{1+r} = k.$$

C.72 Suppose $f: \mathbb{R} \to \mathbb{R}$ is weakly increasing. Suppose $x_n, y_n \in [x^*, \infty)$ s.t. $x_n \to x^*$ and $y_n \to x^*$. Prove that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n).$$

Proof (Harder than exam!)

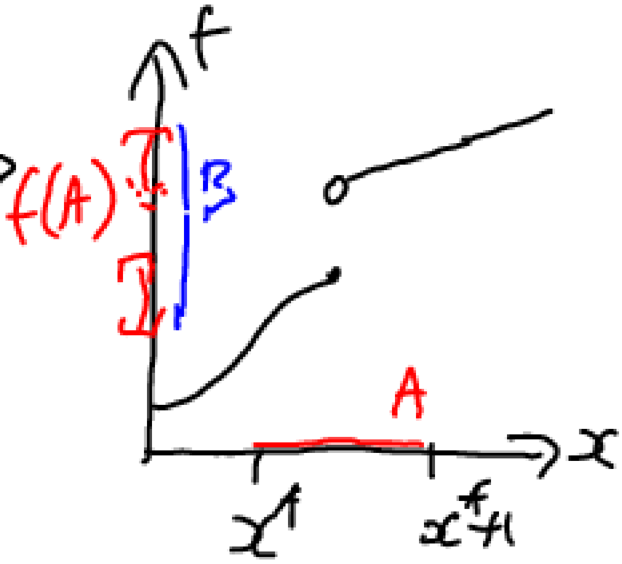
Let $A = [x^*, x^*+1]$.

Without loss of generality, assume that $x_n, y_n \in A$.

Since f is increasing.

$f(A) \subseteq [f(x^*), f(x^*+1)] = B$

Since B is compact, $f(x_n) \in B$ has a convergent subsequence, $f(s_n) \to s^*$.



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Question 1

Household A

consumption c_1^A, c_2^A in periods 1 and 2 ← define notation

$$\max_{\underbrace{c_1^A,\, c_2^A,\, k_1^A,\, k_2^A}_{\text{choices}}} u(c_1^A) + \beta u(c_2^A)$$

s.t. $p_1 c_1^A + p_2 c^A = p_1 k_1^A + p_2 k_2^A$

and $k_1^A + k_2^A = 1$

Market clearing (supply = demand)

$$c_1^A + c_1^B = k_1^A$$ ← first oil market

$$c_2^A + c_2^B = k_2^A + 1$$

$$\boxed{\begin{aligned}\#\,\text{prices} &= \#\,\text{markets} \\ &= \#\,\text{market clearing eq}\end{aligned}}$$

(ii) <u>Social planner's problem</u>

$$\max_{c_1^A,\, c_2^A,\, c_1^B,\, c_2^B} \; u(c_1^A) + u(c_1^B) + \beta u(c_2^A) + \beta u(c_2^B)$$

s.t. $c_1^A + c_1^B \leq 1$

$c_1^A + c_1^B + c_2^A + c_2^B \leq 2.$

Clearly, $c_1^A = c_1^B$ and $c_2^A = c_2^B$.

$$\Rightarrow \max_{c_1,\, c_2} \; u(c_1) + \beta u(c_2)$$

s.t. $c_1 \leq 1 \leftarrow \frac{1}{2}$

$c_1 + c_2 \leq 2. \leftarrow 1 \leftarrow \lambda$

Does the first constraint bind? Solve without, then check

For c_1: $u'(c_1) = \lambda$

c_2: $\beta u'(c_2) = \lambda$

Combining: $u'(c_1) = \beta u'(c_2)$.

$\Rightarrow u'(c_1) < u'(c_2)$

$\Rightarrow c_1 > c_2$

$\Rightarrow c_1 > 2 - c_1$

$\Rightarrow 2c_1 > 2$

$\Rightarrow c_1 > 1$.

$\Rightarrow$ Optimal involves

$c_1 = 1$

$C_2 = 1.$

Question C.72. Suppose $f : \mathbb{R} \to \mathbb{R}$ is weakly increasing. Fix any $x^* \in \mathbb{R}$. Consider any two sequences $x_n, y_n \in [x^*, \infty)$ that both converge to x^*. Prove that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$.

Answer. This is a hard question, that I ought to have broken up into pieces:

Claim 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is weakly increasing. If $x_n \in [x^*, x^* + 1]$ converges to x^*, then $f(x_n)$ is a convergent sequence.

Using this claim, we can prove the result. Without loss of generality, assume that $x_n, y_n \in [x^*, x^* + 1]$. Let $z_1 = x_1$, $z_2 = y_1$, $z_3 = x_2$, $z_4 = y_2$, etc. It is straightforward it prove that $z_n \to x^*$. Applying Claim 1 to z_n, we conclude that $f(z_n)$ is convergent, and converges to some point Z^*. Since $f(x_n)$ and $f(y_n)$ are subsequences of $f(z_n)$, they also converge to Z^*.

Proof of Claim 1: Let $A = [x^*, x^*+1]$ and $B = [f(x^*), f(x^*+1)]$. Since f is weakly increasing, $f(A) \subseteq B$. Since $f(x_n) \in B$ and B is compact, $f(x_n)$ has a convergent subsequence, $f(s_n) \to S^*$.

Now suppose for the sake of contradiction that $f(x_n)$ is not convergent. In a previous question, we established that this implies there must be another convergent subsequence $f(t_n) \to T^* \neq S^*$. Without loss of generality assume that $T^* > S^*$. Let $r = d(S^*, T^*) = T^* - S^*$. Since $f(s_n) \to S^*$, there must be some N such that

$$d(f(s_n), S^*) < r/3 \text{ for all } n \geq N.$$

Since $t_n \to x^*$ and $f(t_n) \to T^*$, there must be some M such that that

$$d(t_n, x^*) < d(t_n, s_N) \text{ and } d(f(t_n), T^*) < r/3 \text{ for all } n \geq M.$$

This implies that $t_M < s_N$ and

$$f(t_M) > T^* - r/3 > S^* + r/3 > f(s_N).$$

This contradicts the condition that f is weakly increasing, so the premise that $f(x_n)$ is non-convergent is mistaken.