Lecture 9

Previously, we assumed the cake goes bad after T days.

What if the cake never goes bad?

$$V_0(k) = \max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(x_t)$$

$$\text{s.t.} \sum_{t=0}^{\infty} x_t = k$$

!

sequence problem

Or, starting day t,

$$V_t(k) = \max_{\{x_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} u(x_s)$$

$$\text{s.t. } \sum_{s=t}^{\infty} x_s = k.$$

Bellman equation:

$$V_t(k) = \max_{x_t,\, k_{t+1} \geq 0} u(x_t) + \beta V_{t+1}(k_{t+1})$$

$$\text{s.t. } x_t + k_{t+1} = k.$$

One Bellman eq for each t.

In fact, $V_0 = V_1 = V_2 = \cdots = V$.

Rewrite Bellman eq:

$$V(k) = \max_{x, k' \geq 0} u(x) + \beta V(k')$$

$$\text{s.t. } x + k' = k.$$

same value function

"recursive" Bellman eq.

$x_{n+1} = f(x_n)$ from Banach's fixed point theorem is also recursive.

<u>Theorem</u> V_0 solves the recursive Bellman eq.

<u>Proof</u>

$$V_0(k) = \max_{\{x_s\}_{s=0}^{\infty}} \sum_{s=0}^{\infty} \beta^s u(x_s) \text{ s.t. } \sum_{s=0}^{\infty} x_s = k$$

$$= \max_{x_0, k_1, \{x_s\}_{s=1}^{\infty}} \sum_{s=0}^{\infty} \beta^s u(x_s) \text{ s.t. } \sum_{s=1}^{\infty} x_s = k_1 \text{ and } x_0 + k_1 = k$$

$$= \max_{\substack{x_0, k_1 \geq 0 \\ \text{s.t. } x_0 + k_1 = k}} \left[\max_{\{x_s\}_{s=1}^{\infty}} \sum_{s=0}^{\infty} \beta^s u(x_s) \text{ s.t. } \sum_{s=1}^{\infty} x_s = k_1 \right]$$

$$= \max_{\substack{x_0, k_1 \geq 0 \\ \text{s.t. } x_0 + k_1 = k}} \left[u(x_0) + \max_{\{x_s\}_{s=1}^{\infty}} \sum_{s=1}^{\infty} \beta^s u(x_s) \text{ s.t. } \sum_{s=1}^{\infty} x_s = k_1 \right]$$

$$= \max_{x_0, k_1 \geq 0} u(x_0) + \beta V_1(k_1)$$

$$\text{s.t. } x_0 + k_1 = k$$

$$= \max_{x, k'} u(x) + \beta V(k')$$

$$\text{s.t. } x + k' = k. \qquad \square$$

Remaining Q's:

* other (wrong) solutions to the Bellman eq?
* maybe V_0 doesn't exist?! (Hilbert's hotel...)
* V increasing, concave, differentiable?
* algorithm to calculate V?

To apply Banach's fixed point theorem, we will rewrite the recursive Bellman as a contraction.

Define <u>Bellman operator</u>:

$$F : CB(\mathbb{R}_+) \longrightarrow CB(\mathbb{R}_+)$$

$$F(V')(k) = \max_{\substack{x, k' \geq 0 \\ \text{s.t. } x + k' = k}} u(x) + \beta V'(k)$$

Finite horizon with T days:

* Once cake is bad $V_{T+1}(k) = u(0)$.
* $V_T = F(V_{T+1})$.
* $V_t = F(V_{t+1})$ for all $t \in \{0, \ldots, T\}$.

Claim: V is a solution to the recursive Bellman eq, if and only if V is a fixed point of the Bellman operator F.

and u is increasing

Lemma (Blackwell) Suppose $u \in CB(\mathbb{R}_+)$. Then the Bellman operator is a contraction of degree β on $(CB(\mathbb{R}_+), d_\infty)$.

Proof

$F(v')(k)$ exists:

Rewrite: $F(v')(k) = \max_{x \in [0,k]} u(x) + \beta V'(k-x)$.

This has a compact domain and a continuous objective (since u and V' are continuous.)
By the Weierstrass ((C.17) Theorem, there is an optimal choice.

$F(V') \in CB(\mathbb{R}_+)$:

Bounded: Since u and V' are bounded (e.g. $\leq \bar{u}$) then

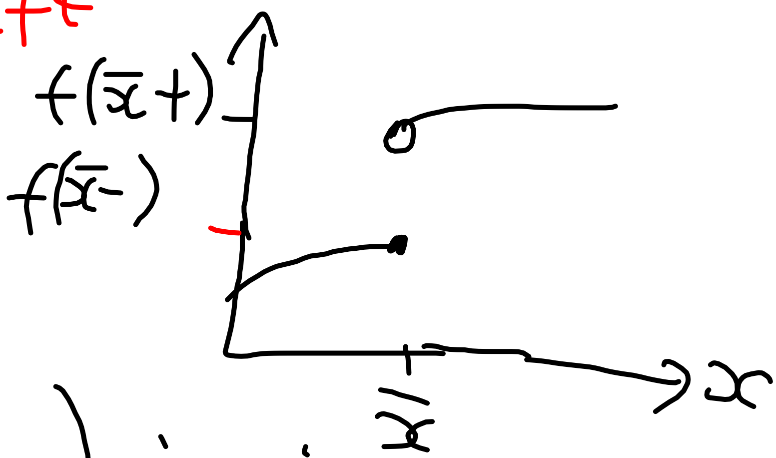
$$F(V')(k) \leq u(k) + \beta V'(k)$$
$$\leq 2\bar{u}.$$

So $F(V')$ is bounded.

Continuous:

Suppose otherwise that

$$\underbrace{F(v')(\bar{k}-)}_{\text{limit from left}} \neq \underbrace{F(v')(\bar{k}+)}_{\text{limit from right}}$$



$F(v')(\cdot)$ is increasing.
(more cake is always better since u is increasing)

$$\Rightarrow F(V')(\bar{k}-) < F(V')(\bar{k}+).$$

Consider the policy,

$$\bar{x}(k) = x(\bar{k}+) - (k - \bar{k}).$$

Since $\bar{x}(k)$ and $u(x)$ and $V'(k)$ are all continuous functions, this policy would give a higher value than $F(V')(\bar{k}-)$.

Conclusion: $F(V') \in CB(\mathbb{R}_+)$

So F is a self-map.

Consider V_1' and V_2'.
Let $x_1(k)$ and $x_2(k)$ be the optimal consumption policies.

$$F(V_1')(k) = u(x_1(k)) + \beta V_1'(k - x_1(k))$$
$$= \left[u(x_1(k)) + \beta V_2'(k - x_1(k))\right] - \beta V_2'(k - x_1(k)) + \beta V_1'(k - x_1(k))$$
$$\leq \left[u(x_2(k)) + \beta V_2'(k - x_2(k))\right] - \beta V_2'(k - x_1(k)) + \beta V_1'(k - x_1(k))$$
$$= F(V_2')(k) - \beta V_2'(k - x_1(k)) + \beta V_1'(k - x_1(k))$$

Therefore,

$$F(V_1')(k) - F(V_2'(k))$$
$$\leq \beta\, d_\infty(V_1', V_2').$$ for all k

Swap V_1' and V_2' and repeat above:

$$F(V_2')(k) - F(V_1'(k))$$
$$\leq \beta\, d_\infty(V_1, V_2')$$ for all k.

Combining:

$$|F(V_1')(k) - F(V_2')(k)|$$
$$\leq \beta\, d_\infty(V_1, V_2)$$ for all k

$$\Rightarrow d_\infty(F(V_1', V_2')) \leq \beta d_\infty(V_1', V_2').$$

We conclude that F is a contraction of degree β. $\square$

We have:

* a complete metric space $(CB(\mathbb{R}_+), d_\infty)$
* a contraction $F: CB(\mathbb{R}_+) \to CB(\mathbb{R}_+)$

So Banach's fixed point theorem says that there is a unique fixed point $V \in CB(\mathbb{R}_+)$

of the Bellman equation, i.e.

$$V = F(V).$$

Answers Q's:

* there are no "wrong" solutions to the recursive Bellman equation.
* is V increasing, concave, etc.?

If A is a closed subset of $CB(\mathbb{R}_+)$ and $F: A \to A$ then the fixed point $V \in A$.

Smaller the set A is, the more you learn about V.

But A can't be too small — need to respect the two requirements of Banach's fixed point theorem.

* Algorithm: $V \approx F^{100}(0)$.

i.e. $F(F(F \ldots (F(\overset{k \mapsto 0}{0})))).$

Example 2.4 (continued)

$$V(p^y; p^r, p^m) = \max_{\substack{r^y, r^x \\ m^y, m^x}} p^y f(g(r^x, m^x), r^y, m^y) - p^r(r^y + r^x) - p^m(m^y + m^x).$$

(ii)

$$\pi(k; p^y; p^r, p^m) = \max_y \; p^y y - C(y; k, p^r, p^m)$$

$$C(y; k, p^r, p^m) = \min_{r^y, m^y} p^r r^y + p^m m^y$$
$$\text{s.t. } f(k, r^y, m^y) \geq y.$$

$$V(p^y; p^r, p^m)$$

$$= \max_{r^x, m^x} \pi(g(r^x, m^x), p^r, p^m) - p^r r^x - p^m m^x.$$