2.4 Dynamic Programming

$$\pi(p;w) = \max_{x \in \mathbb{R}_+^{N-1}} pf(x) - w \cdot x$$

lots of choices!

Bellman eq.

$$= \max_{y \in \mathbb{R}_+} py - c(y;w)$$

state variable

$w \cdot x$

where $\underbrace{c(y;w)}_{\text{cost function}} = \min_{x \in \mathbb{R}_+^{N-1}} w \cdot x$ s.t. $f(x) \geq y$.

what is the cheapest way to meet the output target?

A Bellman equation relates two different value functions.

Bellman equations "bury" (distracting) choices inside a value function.

"Principle of Optimality" refers to any theorem that says Bellman equations give the right answers.

<u>Theorem</u> P. of. O. holds i: the profit maximisation problem. (Lemma 2.1)

<u>Proof</u>

$$\max_{x \in \mathbb{R}_+^{N-1}} pf(x) - w \cdot x$$

$$= \max_{\substack{y \in \mathbb{R}_+, x \in \mathbb{R}_+^{N-1} \\ \text{"choice" s.t. } f(x) = y}} pf(x) - w \cdot x$$

$$= \max_{y \in \mathbb{R}_+} \left(\max_{\substack{x \in \mathbb{R}_+^{N-1} \\ \text{s.t. } f(x) = y}} p\, f(x) - w \cdot x \right)$$

$$= \max_{y \in \mathbb{R}_+} \left(\max_{\substack{x \in \mathbb{R}^{N-1}_+ \\ \text{s.t. } f(x) = y}} py - w \cdot x \right)$$

$$= \max_{y \in \mathbb{R}_+} py + \left(\max_{\substack{x \in \mathbb{R}^{N-1}_+ \\ \text{s.t. } f(x) = y}} -w \cdot x \right)$$

$$= \max_{y \in \mathbb{R}_+} py - \left(\min_{\substack{x \in \mathbb{R}^{N-1}_+ \\ \text{s.t. } f(x) = y}} w \cdot x \right)$$

$$= \max_{y \in \mathbb{R}_+} py - c(y; w). \qquad \square$$

"day 1" "day 2"

Theorem If c is diff. then,

$$p = \left.\frac{\partial c(y;w)}{\partial y}\right|_{y=y(p;w)}$$

marginal cost

Proof FOC of Bellman eq. $\square$

Envelope theorem:

$$\frac{\partial \pi(p;w)}{\partial p} = \left.\left[\frac{\partial}{\partial p}\left(py - c(y;w)\right)\right]\right|_{y=y(p;w)}$$

$$= y(p;w).$$

$$\frac{\partial \pi(p;w)}{\partial w_i} = \left[\frac{\partial}{\partial w_i}\left(py - c(y;w)\right)\right]_{y=y(p;w)}$$

$$= -\left[\frac{\partial c(y;w)}{\partial w_i}\right]_{y=y(p;w)}.$$

Section 3.2 (?)

T # time periods

k_1 cake size at time $t=1$.

x_t cake consumed at time t.

$$V_1(k_1) = \max_{x_1,\dots,x_T \geq 0} u_1(x_1) + \dots + u_T(x_T)$$

$$\text{s.t. } x_1 + \dots + x_T = k_1.$$

Bellman eq

$$\xrightarrow{\text{Bellman eq}} = \max_{x_1, k_2 \geq 0} u_1(x_1) + V_2(k_2)$$

where $V_2(k_2) = \max_{x_2,\dots,x_T \geq 0} u_2(x_2) + \dots + u_T(x_T)$

$\text{s.t. } x_2 + \dots + x_T = k_2.$

Proof of Principal of Optimality:

$$\max_{x_1,\dots,x_T \geq 0} u_1(x_1) + \cdots + u_T(x_T)$$

s.t. $x_1 + \cdots + x_T = k_1$

$$= \max_{x_1,\dots,x_T,\, k_2 \geq 0} u_1(x_1) + \cdots + u_T(x_T)$$

s.t. $x_1 + \cdots + x_T = k_1$, and $x_1 + k_2 = k_1$

$$= \max_{x_1,\, k_2 \geq 0} \left[\max_{x_2,\dots,x_T} u_1(x_1) + \cdots + u_T(x_T) \quad \text{s.t. } x_1 + \cdots + x_T = k_1 \right]$$

s.t. $x_1 + k_2 = k_1$

$$= \max_{\substack{x_1, k_2 \geq 0 \\ \text{s.t. } x_1 + k_2 = k_1}} u_1(x_1) + \left[\max_{\substack{x_2, \ldots, x_T \geq 0 \\ \text{s.t. } x_1 + \cdots = k_1}} u_2(x_2) + \cdots \right]$$

$$= \max_{\substack{x_1, k_2 \geq 0 \\ \text{s.t. } x_1 + k_2 = k_1}} u_1(x_1) + \left[\max_{\substack{x_2, \ldots, x_T \geq 0 \\ \text{s.t. } x_2 + \cdots + x_T = k_2}} u_2(x_2) + \cdots + u_T(x_T) \right]$$

$$= \max_{\substack{x_1, k_2 \geq 0 \\ \text{s.t. } x_1 + k_2 = k_1}} u_1(x_1) + V_2(k_2).$$

$\square$

More generally,

$$V_t(k_t) = \begin{cases} \max_{\substack{x_t, k_{t+1} \\ \text{s.t. } x_t + k_{t+1} = k_t}} u_t(x_t) + V_{t+1}(k_{t+1}) & \text{if } t < T, \\ u_T(k_T) & \text{if } t = T. \end{cases}$$

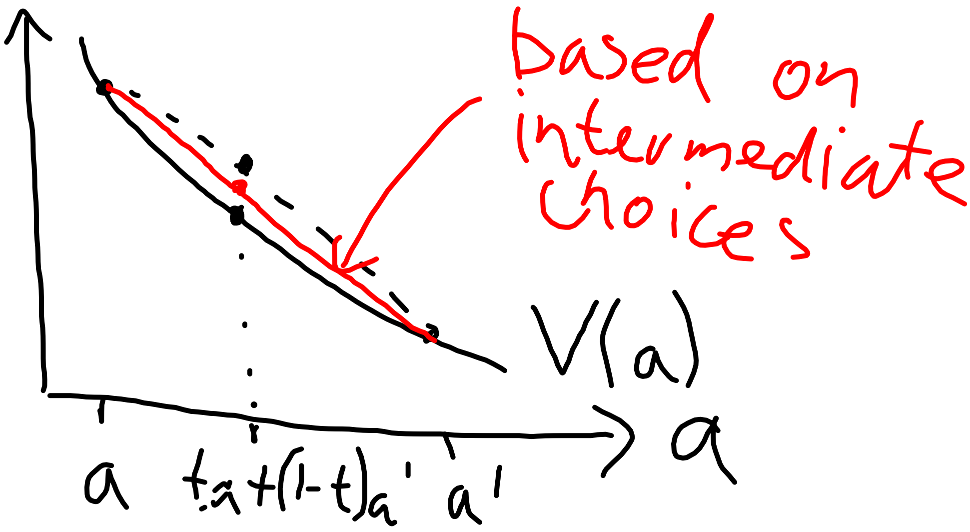
Note:

* P.O.A. is about determining the state variable.
* Mathematically, you can add as many variables as you like to the state.
* Smaller is better.
* I accommodated birthdays, Lent, etc.
* Accommodates $u_t(x_t) = \beta^t u(x)$
i.e. $V(x) = u(x_1) + \beta u(x_2) + \dots + \beta^T u(x_T)$.
("no time period is special")

Theorem 2.6 If v is convex and w is concave, then

$$V(a) = \min_{b} \; v(a,b) \quad \text{s.t.} \quad w(a,b) \geq 0$$

is convex.



Proof

Consider the choice

$$\ell(t) = t\, b(a) + (1-t)\, b(a').$$

the line ----

Then

$$t V(a) + (1-t) V(a')$$

$$= t\, v(a, b(a)) + (1-t)\, v(a', b(a'))$$

red curve

$$\geq v(ta + (1-t)a', \ell(t))$$

(because v is convex)

$$\geq v(ta + (1-t)a', b(ta + (1-t)a'))$$

$\ast$ (since $\ell(t)$ is feasible at $ta + (1-t)a'$, since w is concave)

$= V(ta + (1-t)a').$

Elaborate on $\circledast$:

Will prove that since $w(a, b(a)) \geq 0$ and $w(a', b(a')) \geq 0$ that implies $w(ta + (1-t)a', \ell(t)) \geq 0$.

Now, $w(ta + (1-t)a', \ell(t))$

$= w(ta + (1-t)a, \; tb(a) + (1-t)b(a'))$

$= w(t(a, b(a)) + (1-t)(a', b(a')))$

$$\geq t\, w(a, b(a)) + (1-t)\, w(a', b(a'))$$

(since w is concave)

$$\geq t\, 0 + (1-t)\, 0$$

$$= 0.$$

<u>Theorem</u> If $f : \mathbb{R}_+^{N-1} \to \mathbb{R}_+$ is concave, the cost function is convex in output y.

Proof

Note: not saying $c(y;w)$ is convex in w.

Recall:

$$c(y;w) = \min_{x \in \mathbb{R}^{N-1}_{+}} w \cdot x \quad \text{s.t. } f(x) \geq y.$$

Problem: $g(w,x) = w \cdot x$ is not a convex function.

Think about

$$c(y) = \min_{x \in \mathbb{R}^{N-1}_{+}} \underbrace{w \cdot x}_{\text{linear}} \quad \text{s.t. } f(x) \geq y$$

constant

(convex + concave)

Constraint: $\underbrace{(x,y) \mapsto f(x) - y}_{\text{concave}}$

$\Rightarrow$ Theorem 2.6 implies $c(y)$ is convex. $\square$

Example 2.4

p^R, p^M, p^y prices of raw cocoa and machines and choc bars

k knowledge

(r^x, m^x) cocoa and machines allocated to experimentation

$(r^y, m^y) \Leftarrow$ for prod

$y = f(k, r^y, m^y)$ choc bar output

$K = g(r^x, m^x)$ knowledge output.

$$V(p^y; p^r, p^m)$$

$$= \max_{r^y, m^y, r^x, m^x} p^y f(g(r^x, m^x), r^y, m^y) - p^r(r^x + r^y) - p^m(m^x + m^y)$$