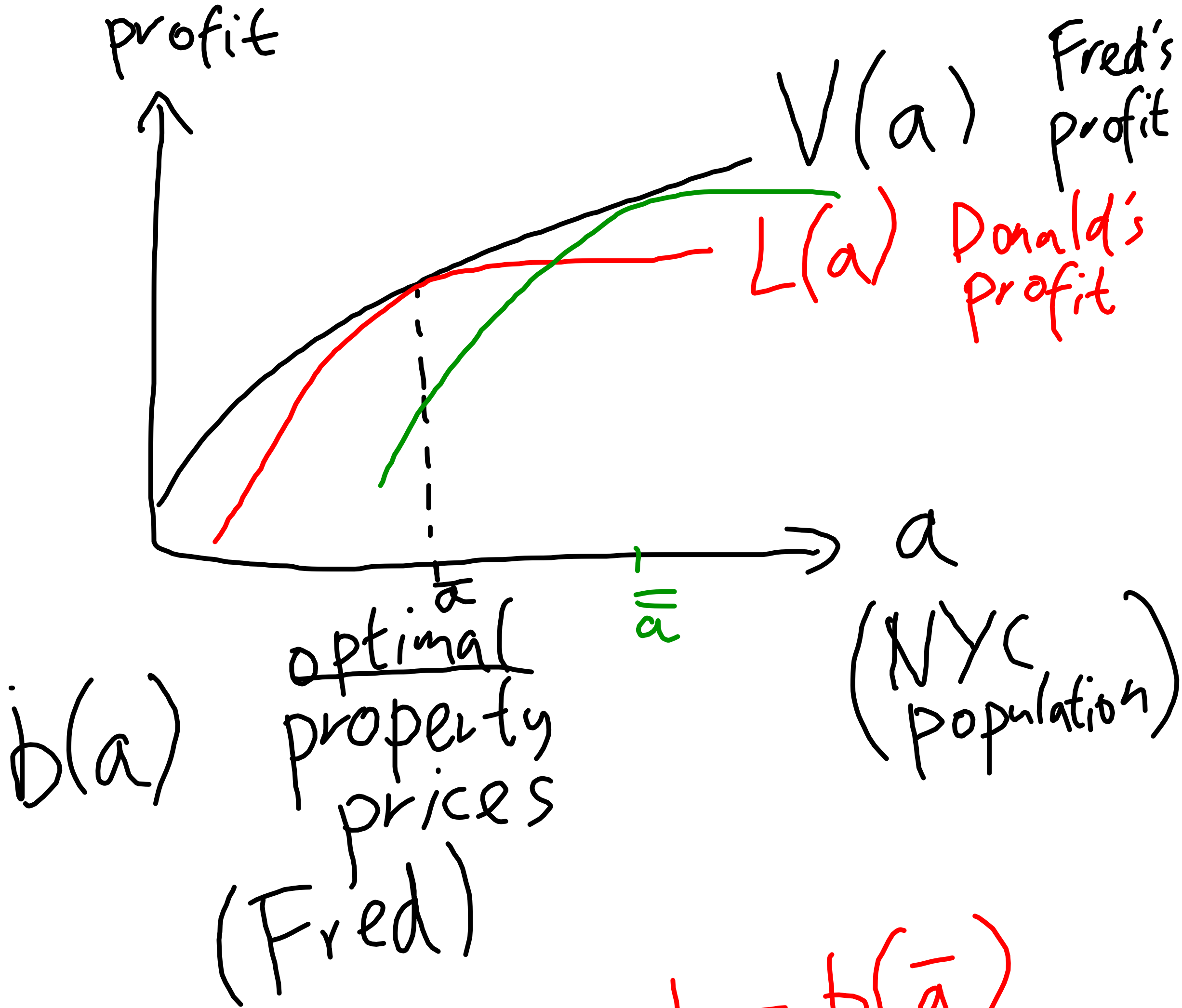
$$V(a) = \max_{b} v(a, b)$$

Theorem If v is differentiable and V is differentiable, then

$$V'(a) = \left.\frac{\partial v(a,b)}{\partial a}\right|_{b=b(a)}$$

$$= v_a(a, b(a)).$$



Final words: $b = b(\bar{a})$

Assume Donald picks $b(\bar{a})$ for all a. $L(a) = v(a, b(\bar{a}))$.

So $L(a) \leq V(a)$ for all a

$L(\bar{a}) = V(\bar{a})$.

$L'(a) = v_a(a, b(\bar{a}))$.

$L'(\bar{a}) = v_a(\bar{a}, b(\bar{a}))$.

Since $\bar{a}$ solves

$$\min_a V(a) - L(a).$$

$\Rightarrow$ FOC's: $V'(\bar{a}) = L'(\bar{a})$.

We conclude $V'(\bar{a}) = v_a(\bar{a}, b(\bar{a}))$ for all $\bar{a}$.

$\square$

e.g. w wage
ℓ labour
profit function

$$\pi(w) = \max_{\ell} 10\sqrt{\ell} - w\ell$$

What is $\pi'(w)$?

Method 1 (w/o envelope theorem)

1. Solve $\ell(w)$. FOC's:

$$\frac{5}{\sqrt{\ell}} - w = 0$$

$$\frac{5}{\sqrt{l}} = w$$

$$\frac{\sqrt{l}}{5} = \frac{1}{w}$$

$$\sqrt{l} = \frac{5}{w}$$

$$l(w) = \frac{25}{w^2}.$$

got rid of max

2. Substitute

$$\pi(w) = 10\sqrt{l(w)} - w\,l(w)$$

$$= 10\left(\frac{5}{w}\right) - w\left(\frac{25}{w^2}\right)$$

$$= \frac{50}{w} - \frac{25}{w}$$

$$= \frac{25}{w}.$$

3. <u>Diff.</u>

$$\pi'(w) = -\frac{25}{w^2}.$$

$$- \ell(w)$$

Method 2 (w/ envelope theorem)

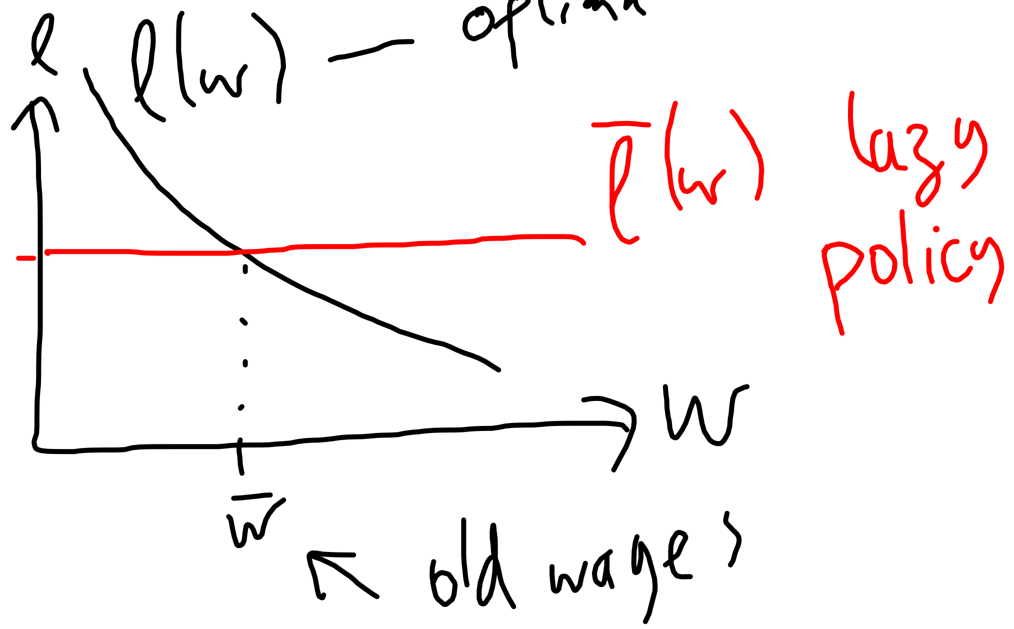
$$\pi'(w) = \left[\frac{\partial}{\partial w}\left(10\sqrt{l} - wl\right)\right]_{l=l(w)}$$

$$= \left[-l\right]_{l=l(w)}$$

$$= -l(w).$$

Instead of the optimal policy $\ell(w)$, we could imagine a lazy firm that uses the policy

$$\bar{\ell}(w) = \ell(\bar{w})$$



lazy firm's profit funtion

$$L(w) = 10\sqrt{\bar{\ell}(w)} - w\,\bar{\ell}(w)$$

$$L'(w) = -\bar{\ell}(w) \quad \left(\text{since } \bar{\ell}'(w) = 0\right)$$

$$\pi(p;w) = \max_{x \in R_+^{N-1}} pf(x) - w\cdot x$$

$$\frac{\partial \pi(p;w)}{\partial p} = \frac{\partial}{\partial p}\Big[pf(x) - w\cdot x\Big]\Bigg|_{x = x(p;w)}$$

$$= f(x)\Big|_{x = x(p;w)}$$

$$= f(x(p;w))$$

$$= y(p;w) \leftarrow \text{output policy}$$

$$\frac{\partial \pi(p,w)}{\partial w_i} = \frac{\partial}{\partial w_i}\left[pf(x) - w\cdot x\right]\Bigg|_{x=x(p,w)}$$

$$= \left[-x_i\right]_{x=x(p,w)}$$

$$= -x_i(p,w).$$

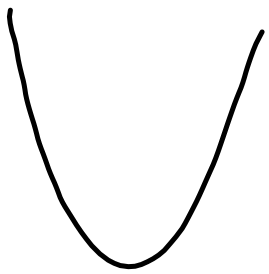
Differentiating again:

$$\frac{\partial^2 \pi(p, w)}{\partial p^2} = \frac{\partial y(p, w)}{\partial p}$$

$$\frac{\partial^2 \pi(p, w)}{\partial w_i \, \partial w_j} = -\frac{\partial x_i(p, w)}{\partial w_j}$$

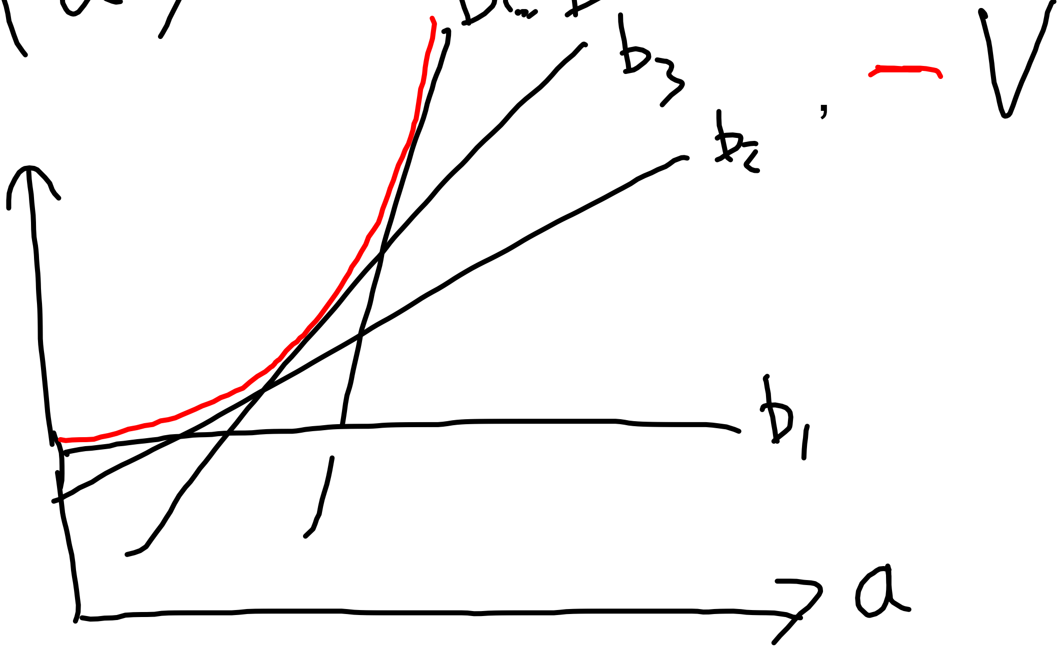
$$= \frac{\partial^2 \pi(p, w)}{\partial w_j \, \partial w_i} = -\frac{\partial x_j(p, w)}{\partial w_i}$$

i = engineers
j = lawyers.



convex

$$V(a) = \max_{b \in B} v(a,b)$$



<u>Theorem</u> If each $v(\cdot, b)$ is convex, then V is convex.

<u>Proof</u> We want to prove

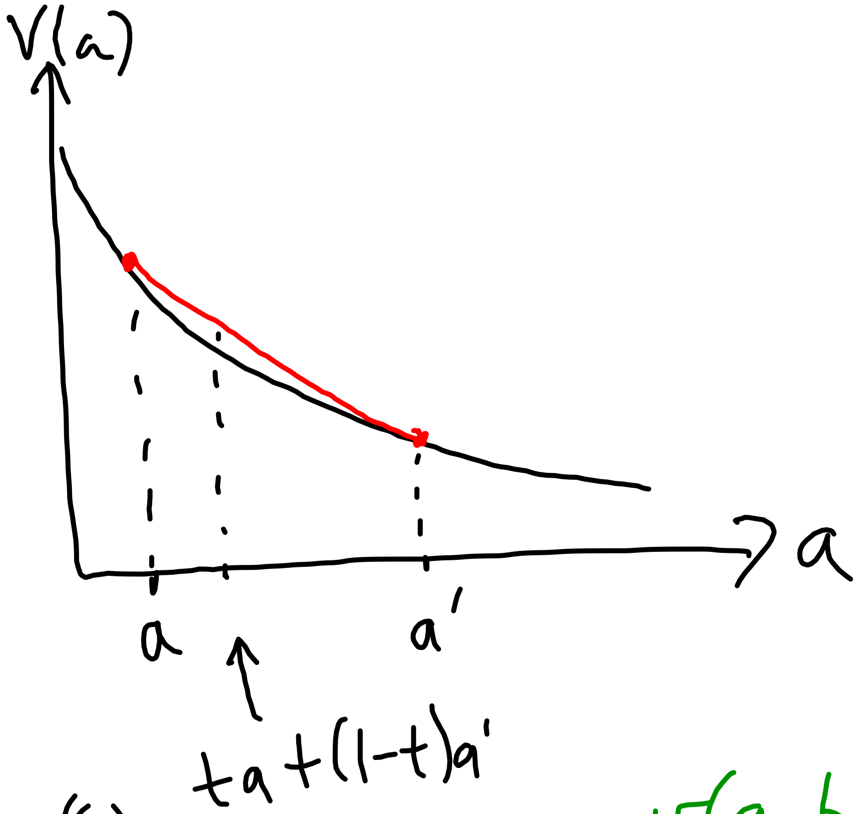
line

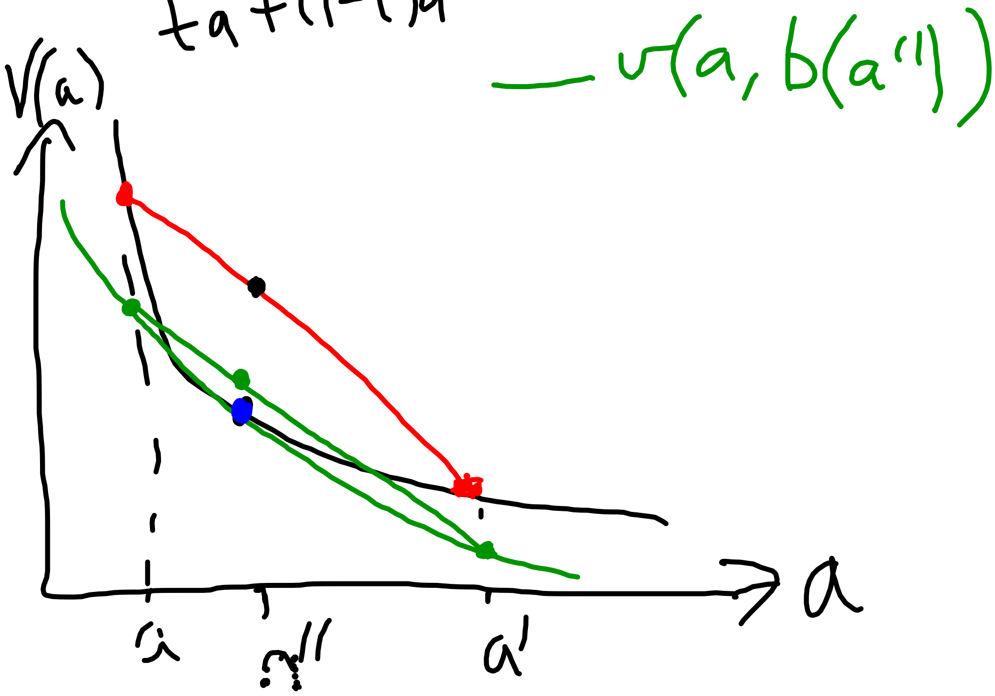
$$tV(a)+(1-t)V(a')$$

curve

$$\geq V(ta+(1-t)a')$$

for all $t \in [0,1]$ and $a, a' \in A$


V(a)
a
a
a'
ta+(1-t)a'


V(a)
— v(a, b(a''))
a
a
a''
a'

top

$$\begin{cases} tV(a)+(1-t)V(a') \\ = t\,v(a, b(a)) + (1-t)\,v(a, b(a')) \end{cases}$$

$$\geq t\,v(a, b(a'')) + (1-t)\,v(a, b(a'))$$

where $a'' = ta + (1-t)a'$,

$$\geq t\,v(a, b(a'')) + (1-t)\,v(a, b(a''))$$

$$\geq v(ta + (1-t)a', b(a''))$$

$$= v(a'', b(a''))$$

$$= V(a'') = V(ta+(1-t)a').$$

$\square$

<u>Theorem</u> For every production function $f: \mathbb{R}_+^{N-1} \to \mathbb{R}$, the firm's profit function

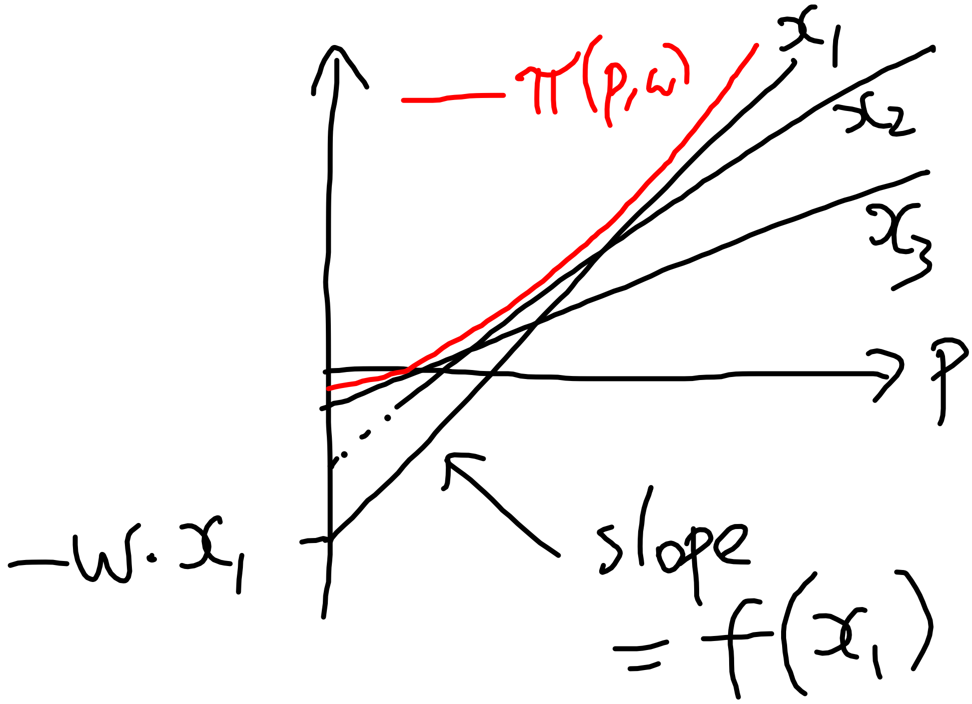
$$\pi(p; w) = \max_x pf(x) - w \cdot x$$

is convex.

<u>Proof</u> For each input choice x, the function

$$(p, w) \mapsto pf(x) - w \cdot x$$

is linear, and hence convex.



By the previous theorem, V is a convex function.

$\square$

<u>Theorem</u> For every production function $f: \mathbb{R}_+^{N-1} \to \mathbb{R}$, if the profit function is smooth (twice diff), then

$$\frac{\partial y(p,w)}{\partial p} \geq 0$$

and $$\frac{\partial x_i(p,w)}{\partial w_i} \leq 0.$$

Proof

By the envelope theorem,

$$\underbrace{\frac{\partial \pi^2(p;w)}{\partial p^2}}_{\geq 0} = \frac{\partial y(p;w)}{\partial p}.$$

Since π is convex (by previous theorem), the left side ≥ 0.

$\Rightarrow$ right side ≥ 0. $\square$