2.1 Production functio s

N goods

1 output

$N-1$ inputs

$$\left.\begin{aligned} & f : \mathbb{R}^{N-1}_{+} \longrightarrow \mathbb{R}_{+} \\ & \underset{\text{output}}{y} = f(\underset{\text{inputs}}{x}) \end{aligned}\right\} \text{production function}$$

possible assumptions:

$f(0) = 0$

$\in \mathbb{R}^{N-1}$

i.e. $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

possibility of inaction

free disposal (or monotonicity).

If $x \leq x'$, i.e. $x_n \leq x_n'$ for all n, then $f(x) \leq f(x')$.

Smoothness f is twice differentiable. Each partial derivative

$$\frac{\partial f}{\partial x_i}$$

is called the marginal productivity of good x_i.

decreasing marginal productivity

$\frac{\partial f(x)}{\partial x_i}$ is decreasing in x_i.

weakly increasing returns to scale

For all $x \in \mathbb{R}^{N-1}_+$, and all scales $t > 1$:

$$f(tx) \geq t f(x).$$

constant returns to scal

For all $x \in \mathbb{R}_+^{N-1}$ and all $t > 0$,

$$f(tx) = t f(x).$$

weakly decreasing returns to scale

For all $x \in \mathbb{R}_+^{N-1}$ and all $t > 1$,

$$f(tx) \leq t f(x).$$

Concavity

f is a concave function.

detour: A.3 (Geometry)

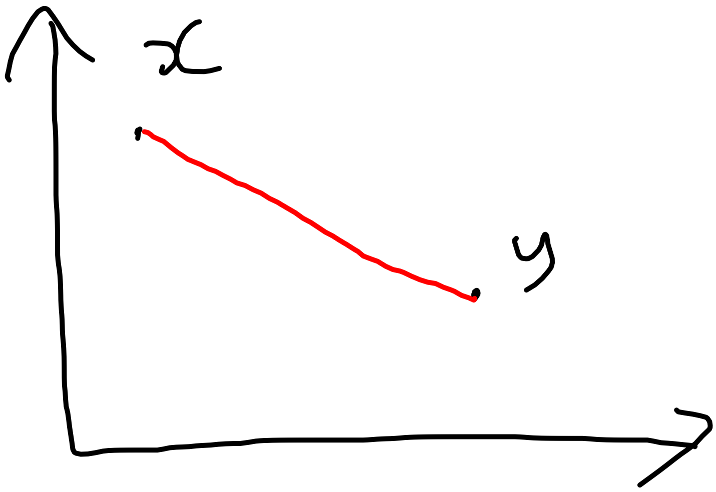
Def ~~A~~ The closed interval

convex combination

between $x, y \in \mathbb{R}^n$ is

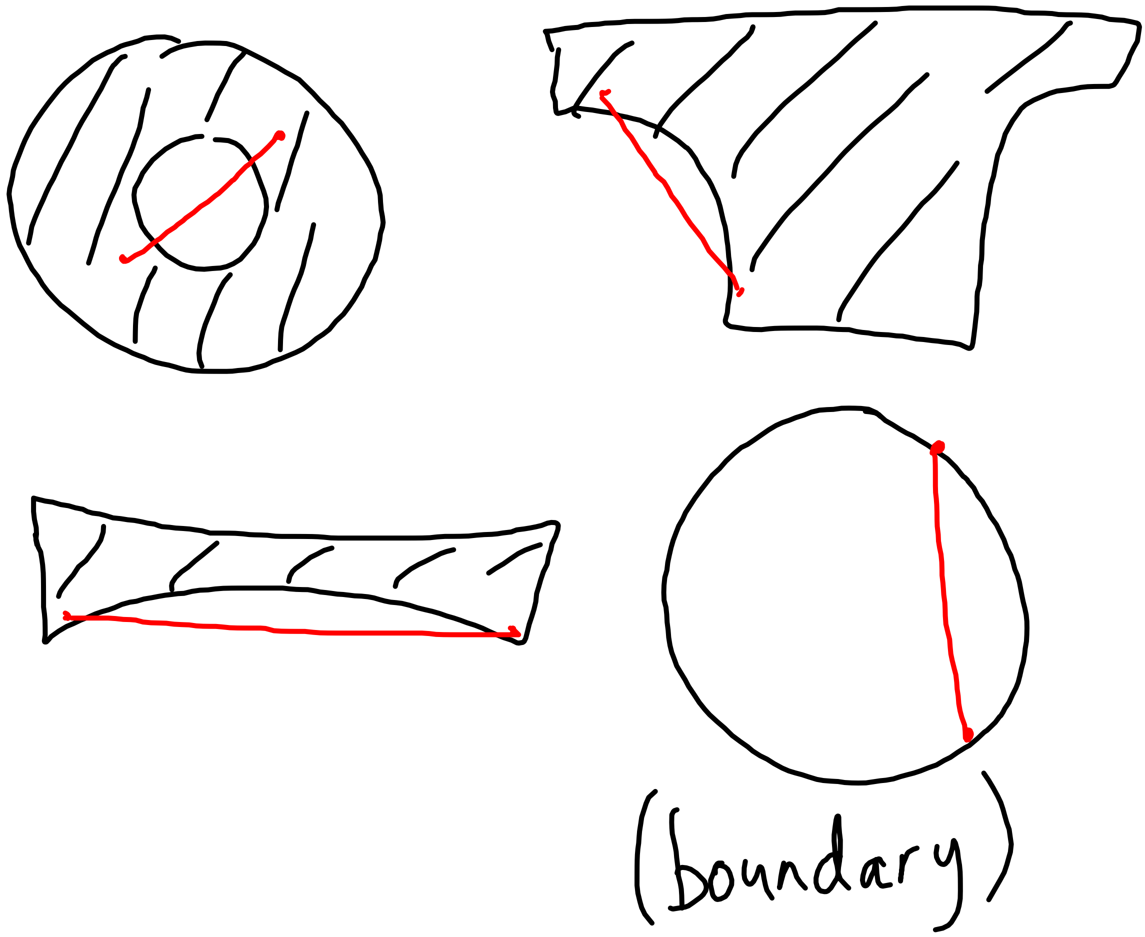
$$[x,y] = \{ax + (1-a)y : a \in [0,1]\}$$

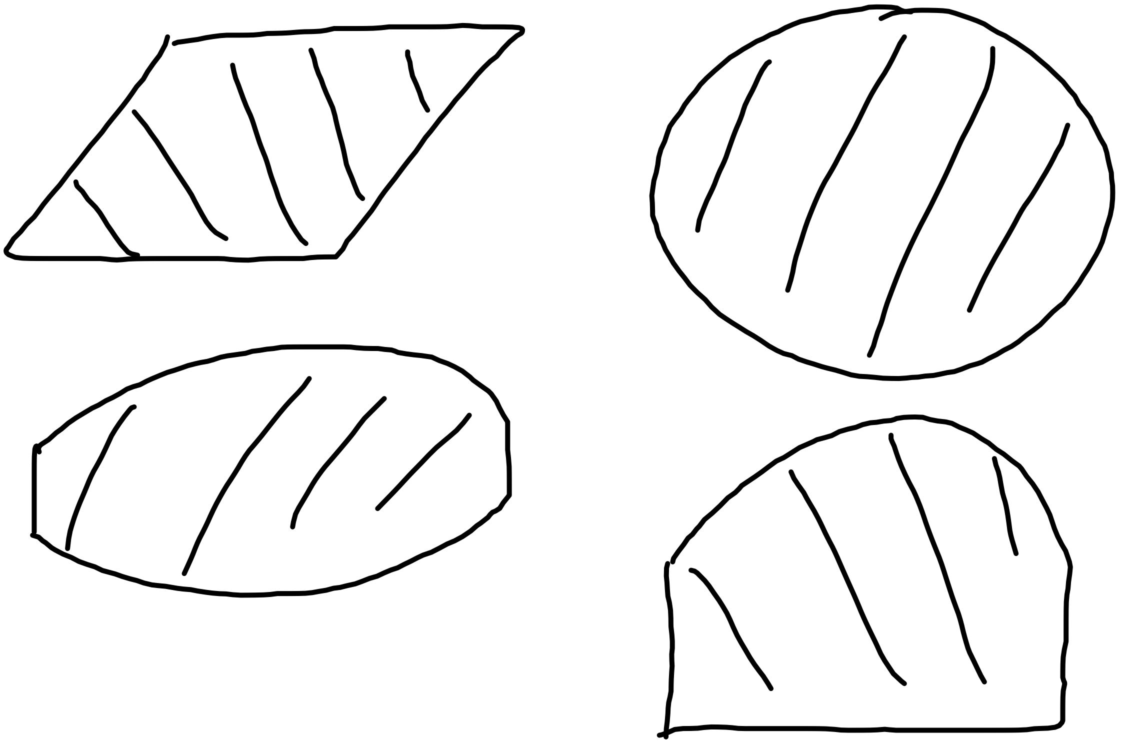
Similar definitions for $[x,y)$, (x,y), etc.



<u>Def</u> A set $X \subseteq \mathbb{R}^n$ is <u>convex</u> if for all $x, y \in X$, the interval $[x, y]$ is contained in X, i.e.

$$[x, y] \subseteq X.$$


(boundary)



<u>Theorem</u> The intersection of convex sets is convex.

<u>Proof</u> Suppose A and B are convex sets. We need to show that if $x, y \in A \cap B$ then $[x, y] \subseteq A \cap B$.

If $x, y \in A \cap B$ then $x, y \in A$.

Since A is convex,

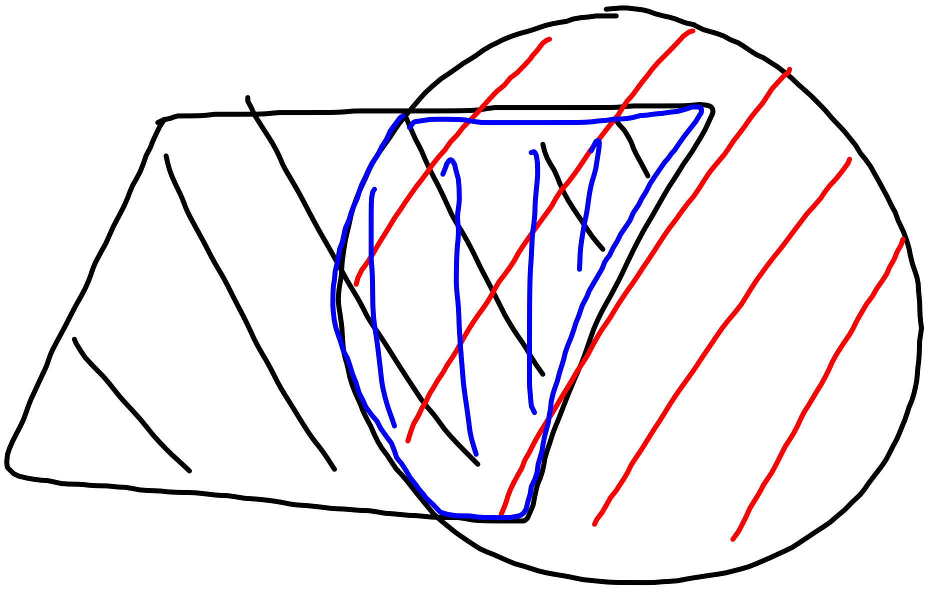
$$[x, y] \subseteq A.$$

By the same logic,

$$[x, y] \subseteq B.$$

Therefore $[x, y] \subseteq A \cap B$.

We conclude that $A \cap B$ is a convex set. $\square$

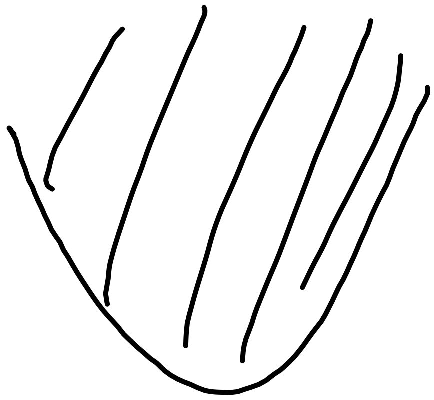


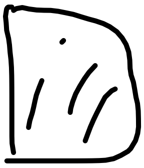
Def $f: X \to \mathbb{R}$ is a **convex function** if its **hypergraph**

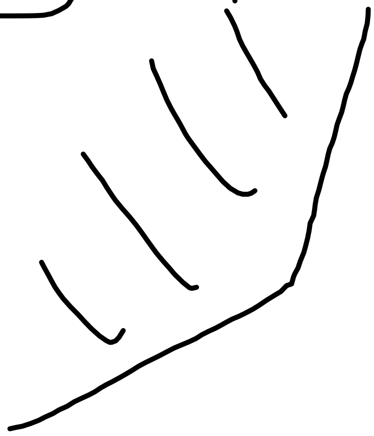
$$\{(x,y) : x \in X,\ y \geq f(x)\}$$

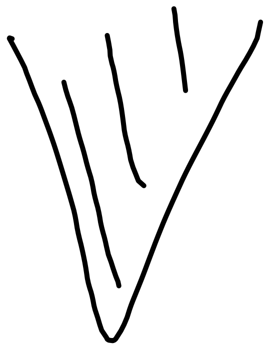
is a convex set.

Note: a function can only be convex if its domain is convex.



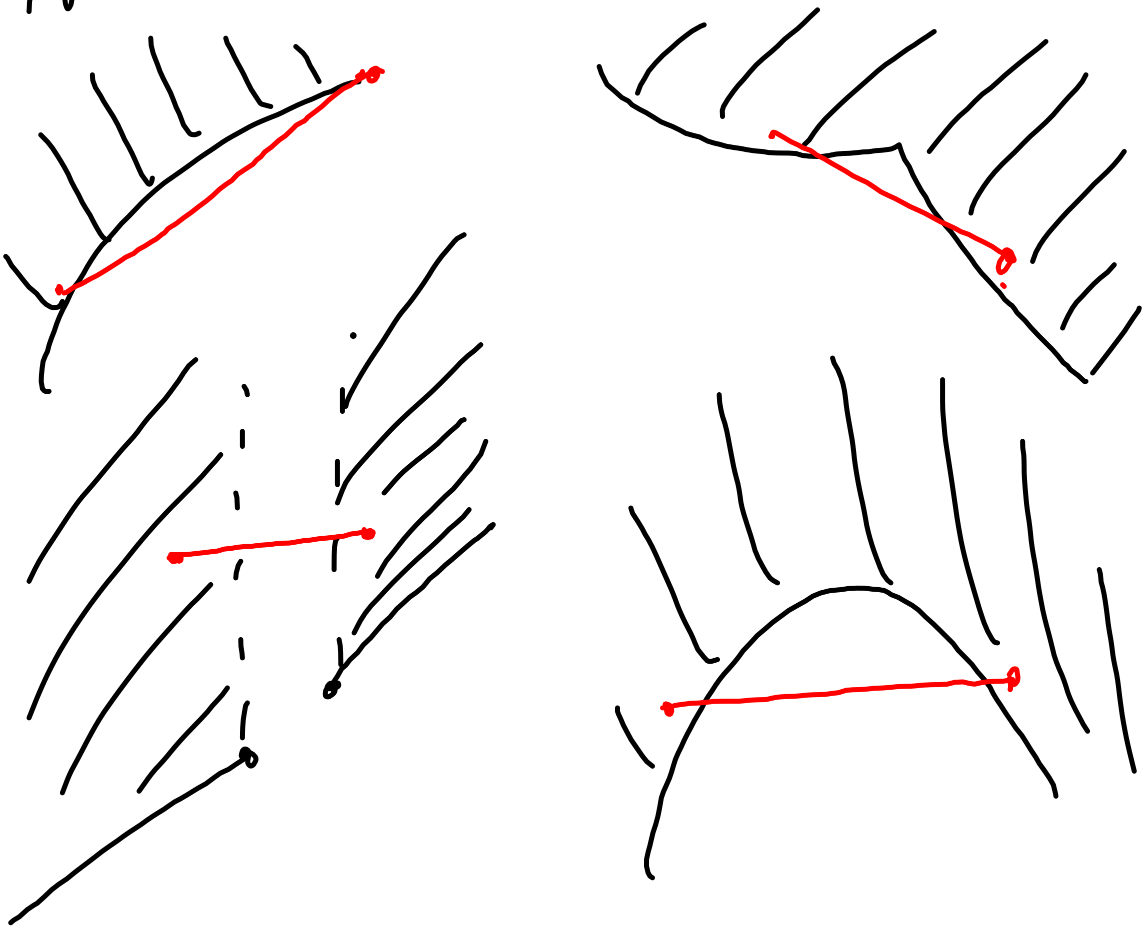
 hypergraph





are convex functions

NOT convex:

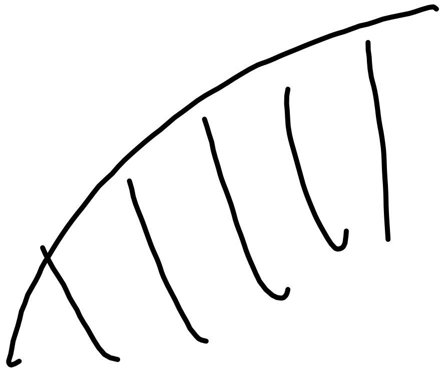


<u>Def</u> $f : X \to \mathbb{R}$ is a <u>concave function</u> if its hypograph

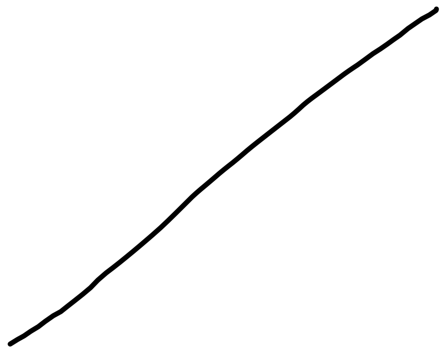
$$\{(x, y) : x \in X, \ y \leq f(x)\}$$

is a convex set.

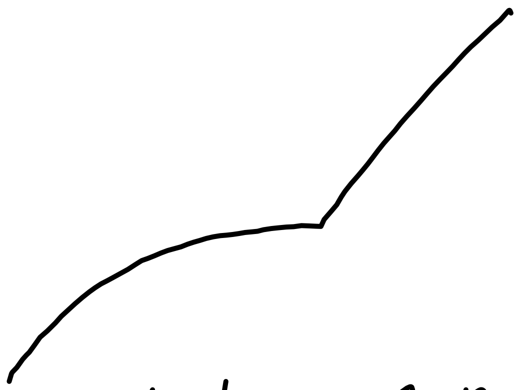
<u>Theorem</u> f is a concave function if and only if $-f$ is a convex function.



concave



convex and concave

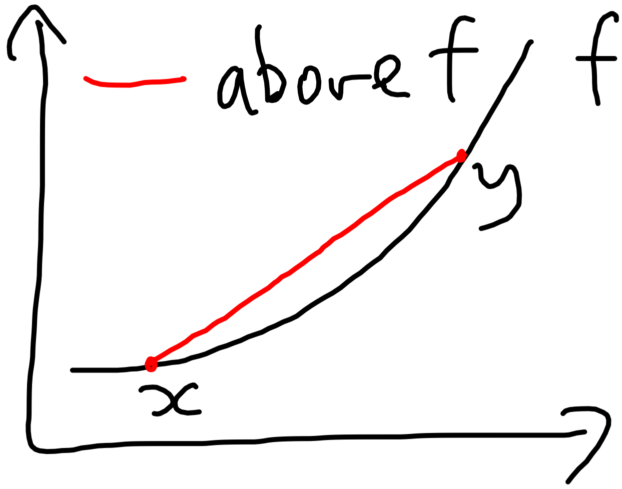


neither convex nor concave

<u>Theorem</u> If $f: X \to \mathbb{R}$ is a convex function and X is an open set in $(\mathbb{R}^n, d_2)$, then f is continuous.

<u>Theorem</u> Suppose $f: \mathbb{R} \to \mathbb{R}$ is differentiable. Then f is a convex function if and only if its derivative f' is weakly increasing.

Theorem Suppose $f:\mathbb{R}\to\mathbb{R}$ is twice differentiable. Then f is convex if and only if $f''(x)\geq 0$ for all $x\in\mathbb{R}$.



<u>Theorem</u> A function $f: X \to \mathbb{R}$ is convex if and only if for all $x, y \in X$, and $a \in (0,1)$

$$\underbrace{af(x) + (1-a)f(y)}_{\text{line}} \geq \underbrace{f(ax + (1-a)y)}_{\text{curve}}$$

Back to production functions.

Assumption f is a concave function.

and smooth

Claim If f is concave, then f has decreasing marginal productivity.

Claim If f is smooth, concave, and has the possibility of inaction, then f has weakly decreasing returns to scale.

Proof

We want to show

$$f(tx) \leq tf(x)$$

for all $x \in \mathbb{R}_+^{N-1}$ and all $t > 1$.

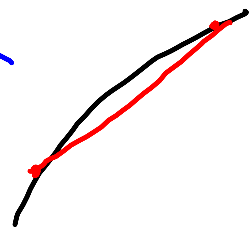
Let $a = \frac{1}{t}$. So $a \in (0,1)$.

We will apply the theorem to the points $0 \in \mathbb{R}^{N-1}$ and tx with $a = \frac{1}{t}$.

weight

$= 0$

$$\underbrace{a f(tx) + \cancel{(1-a) f(0)}}_{\text{line}}$$



$$\leq f(atx + \cancel{(1-a)0})$$

$$\Rightarrow \quad a\, f(tx) \leq f(atx).$$

$$\Rightarrow \quad \frac{1}{t} f(tx) \leq f\left(\cancel{\frac{1}{t}}\, tx\right)$$

$$\Rightarrow \quad \frac{1}{t} f(tx) \leq f(x)$$

$$\Rightarrow \quad f(tx) \leq t f(x) \ \square$$

upper contour sets are convex (another assumption)

f is concave ⇗

(skipped)

Compact $\Rightarrow$ closed, bounded

any metric space

closed, bounded

$\Rightarrow$ compact

Only in $(\mathbb{R}^n, d_2)$.

If (X,d) and (Y,d) are metric spaces and $K \subseteq X \cap Y$, then if K is compact in X, then K is also compact in Y.

<u>Proof</u>

K is compact in X

$\Longleftrightarrow$ every $x_n \in K$

has a subsequence y_n

s.t. $y_n \xrightarrow{d} y^* \in K$

$\Longleftrightarrow$ K is compact in Y.

$\square$

2.2 Profit maximiation

$p > 0$ output price

$w \in \mathbb{R}_+^{N-1}$ input prices

(w including wages, raw materials, parts, office rent)

$$\underbrace{\pi(p; w)}_{\text{profit function}} = \max_{x \in \mathbb{R}_+^{N-1}} pf(x) - w \cdot x$$

$$= pf(x(p;w)) - w \cdot \underbrace{x(p;w)}_{\text{factor demand}}$$

F.O.C. for optimal x_i:

$$p \frac{\partial f(x)}{\partial x_i} = w_i$$

FOC x_i / x_j:

$$\frac{\dfrac{\partial f(x)}{\partial x_i}}{\dfrac{\partial f(x)}{\partial x_j}} = \frac{w_i}{w_j}.$$

Example 2.1

r royalties for each song

l_m musician input

l_t technician input

$\left.\begin{matrix} w_m \\ w_t \end{matrix}\right\}$ wages

$f(l_m, l_t)$ output of songs

$$\pi(r; w_m, w_t) = \max_{l_m, l_t} r f(l_m, l_t) - w_m l_m - w_t l_t$$

Example 2.2

w waste input

$g(w)$ glycerine output

$d(w)$ diesel output

p^w, p^g, p^d prices of waste, glycerine, diesel

$$\pi(p^g, p^d; p^w) = \max_w p^g g(w) + p^d d(w) - p^w w.$$

Example 2.3

x crude oil input

$e = f(x)$ ethylene output

E " allocd to plastic

$y = g(E)$ plastic output

p^x, p^e, p^y prices of oil, ethylene, plastic

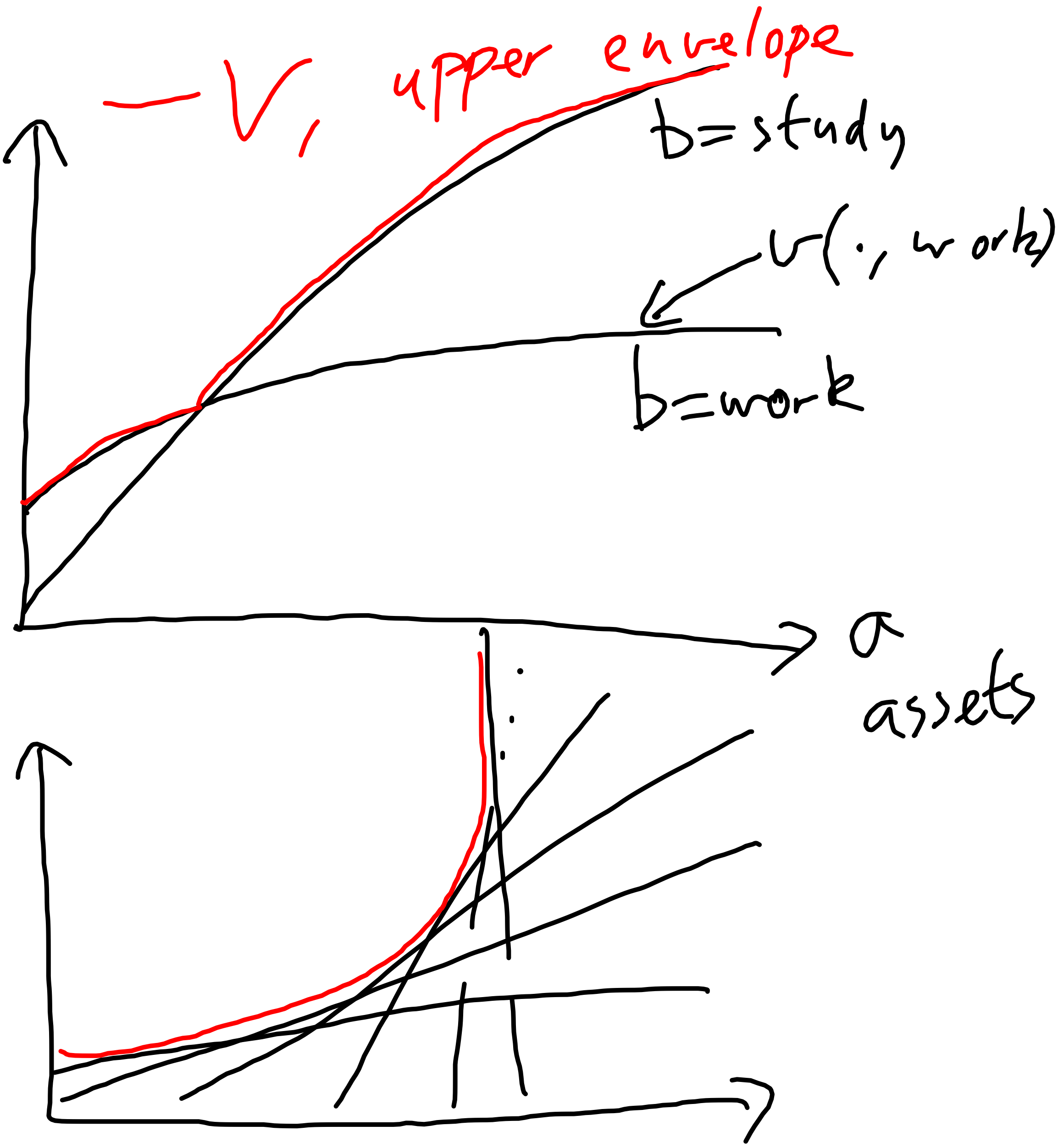
$$\pi(p^x, p^e, p^y) = \max_{x, E} \; p^e\big(\underbrace{f(x) - E}_{\text{ethylene sold}}\big) + p^y g(E) - p^x x.$$

2.3 Upper envelopes and value functions

$$\underbrace{V(\underbrace{a}_{\text{state variable}})}_{\text{value function}} = \max_{b \in B} \underbrace{v(a, b)}_{\text{objective function}}$$

choice variable

choice set



— V, upper envelope
b=study
v(·, work)
b=work
a
assets

<u>Theorem</u> Let $v : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a differentiable function,

$$V(a) = \max_{b \in \mathbb{R}^m} v(a, b)$$

and $b(a) \in \underset{b \in \mathbb{R}^m}{\arg\max}\, v(a, b)$.

If V is differentiable at a, then

$$V'(a) = \left. \frac{\partial v(a, b)}{\partial a} \right|_{b = b(a)}$$

or $V'(a) = v_a(a, b(a))$.

Note: could have tried to differentiate $a \mapsto v(a, b(a))$, which would involve $b'(a)$.