eg: $(X, d) = ((0,1], d_2)$

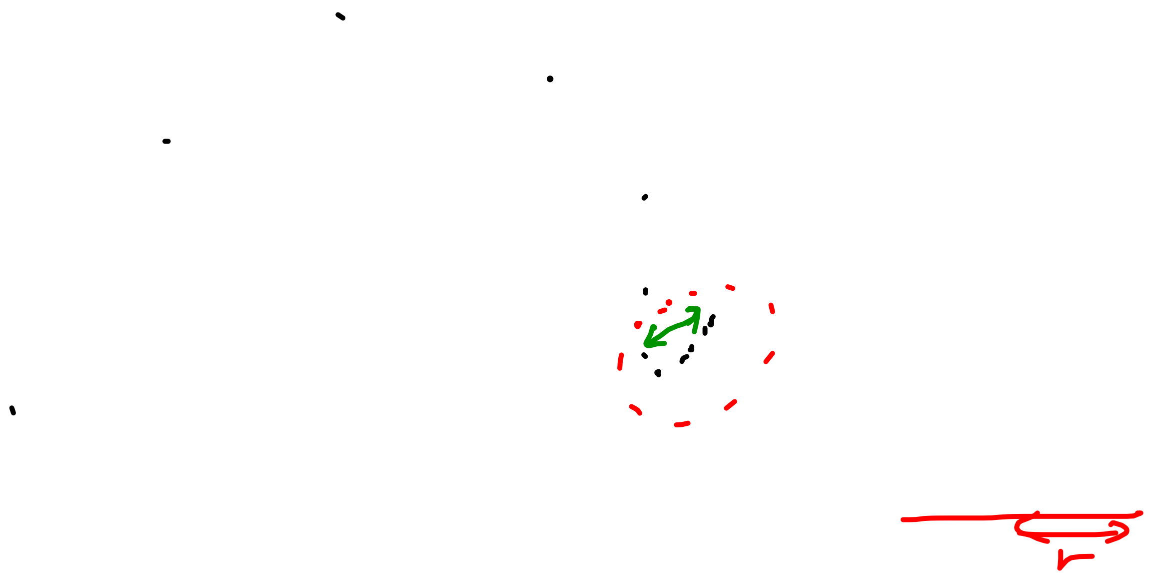
$x_n = \frac{1}{n}$

Wants to converge to 0.

But $0 \notin X$.

<u>Def</u> Let (X, d) be a metric space. A sequence $x_n \in X$ is a <u>Cauchy sequence</u> if for every radius $r > 0$, there exists some N s.t. for all $n, m > N$,

$$d(x_n, x_m) < r.$$



<u>Def</u> A metric space (X,d) is <u>complete</u> if every Cauchy sequence is convergent.
Specifically, if $x_n \in X$ is a Cauchy sequence, then there is some point $x^* \in X$ such that

$$x_n \to x^*.$$

* $(\mathbb{R}, d_2)$ is a complete metric space.

* $(\mathbb{Q}, d_2)$ is <u>not</u> complete.

$$x_1 = 3$$
$$x_2 = 3.1$$
$$x_3 = 3.14$$
$$x_4 = 3.141$$
$$\vdots$$

x_n "wants" to converge to π, but $\pi \notin \mathbb{Q}$.

x_n is not a convergent sequence.

* $((0,1], d_2)$ is <u>not</u> complete.

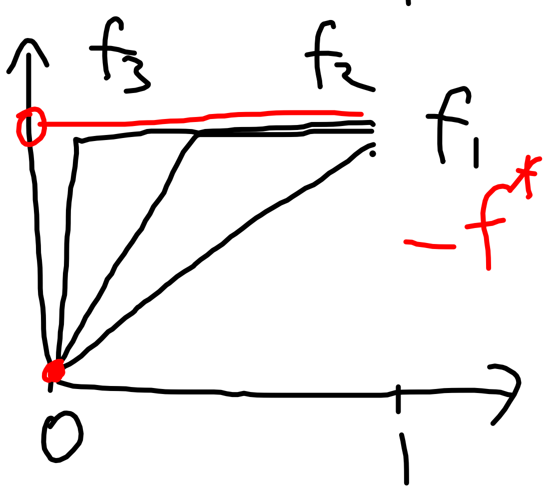
$x_n = \frac{1}{n}$ is a Cauchy seq,

but x_n is not convergent.

* $(C([0,1]), d)$, continuous functions $[0,1] \to \mathbb{R}$

$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx.$$

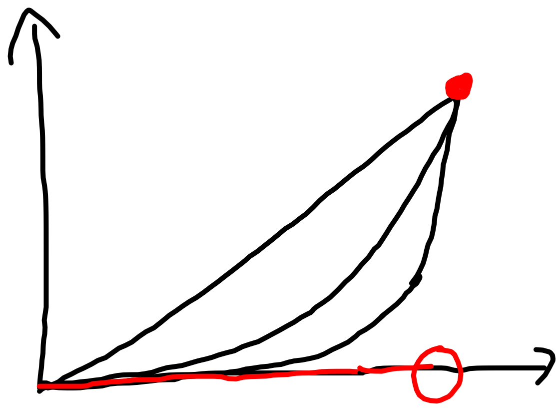
is <u>not</u> complete.



f_n wants to converge to f^*.

But $f^* \notin C([0,1])$.

$$f_n(x) = x^n$$



I will double-check.

Theorem Let (X, d) be a metric space. If $x_n \to x^*$ then x_n is a Cauchy sequence.

Proof

Fix any radius $r > 0$. Since x_n is convergent, there is some N such that

$$d(x_n, x^*) < \frac{r}{2} \text{ for all } n > N.$$

By the triangle inequality,

$$d(x_n, x_m) \leq \underbrace{d(x_n, x^*)}_{< \frac{r}{2}} + \underbrace{d(x_m, x^*)}_{< \frac{r}{2}}$$

If $n, m > N$, then

$$\Rightarrow d(x_n, x_m) < r.$$

$\square$

Theorem If $x_n \in X$ is Cauchy and $y_n \to y^*$ is a subsequence of x_n, then $x_n \to y^*$.

Proof

Pick any radius $r > 0$.

Since x_n is Cauchy, there is

some N such that

$$d(x_n, x_m) < \frac{r}{2} \text{ for all } n, m > N.$$

Since $y_n \to y^*$, there is some $k > N$ such that

$$d(y_k, y^*) < \frac{r}{2}.$$

Pick m such that $x_m = y_k$.

$$\Rightarrow d(x_m, y^*) < \frac{r}{2}.$$

Therefore:

$$\begin{aligned} d(x_n, y^*) &\leq d(x_n, y_k) \\ &\quad + d(y_k, y^*) \\ &< \frac{r}{2} + \frac{r}{2} \\ &= r \quad \text{for all } n > N. \end{aligned}$$

Therefore, $x_n \to y^*$. $\square$

<u>Theorem</u> If $x_n \in X$ is Cauchy, then x_n is bounded.

<u>Proof</u>:

Recall: x_n is bounded if there is some radius $r>0$ such that $d(x_0, x_n) < r$ for all n.

Pick $r=1$.

Since x_n is Cauchy,

there is some N s.t.

$$d(x_n, x_m) < 1 \text{ for all } n, m > N$$

By the triangle inequality

$$d(x_0, x_n)$$
$$\leq d(x_0, x_{N+1}) + d(x_{N+1}, x_n)$$
for all n
$$< d(x_0, x_{N+1}) + 1$$
for all n. $\square$

r we need.

Theorem If x_n is Cauchy, and y_n is a subsequence, then y_n is Cauchy.

Theorem Let (X, d_X) and (Y, d_Y) be metric spaces. If (Y, d_Y) is complete, then

(i) $(B(X,Y), d_\infty)$ is complete

(ii) $(CB(X,Y), d_\infty)$ is complete.

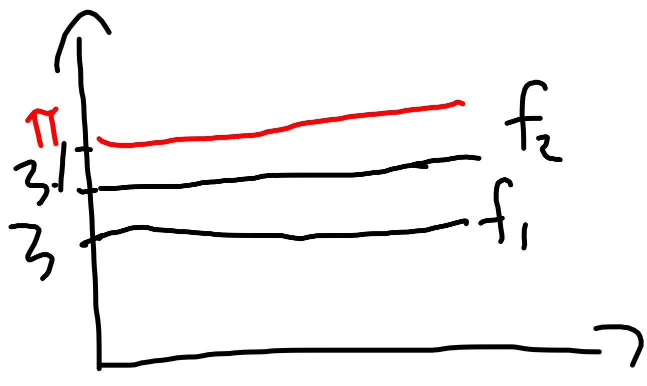
$$B(X,Y) = \{ f : X \to Y,$$

such that f is bounded$\}$

$$CB(X,Y) = \{ f \in B(X,Y) :$$

f is continuous$\}$.

$$d_\infty(f,g) = \sup_{x \in X} d_Y(f(x), g(x)).$$



$f^*(x) = \pi$.

co-domain must be complete.

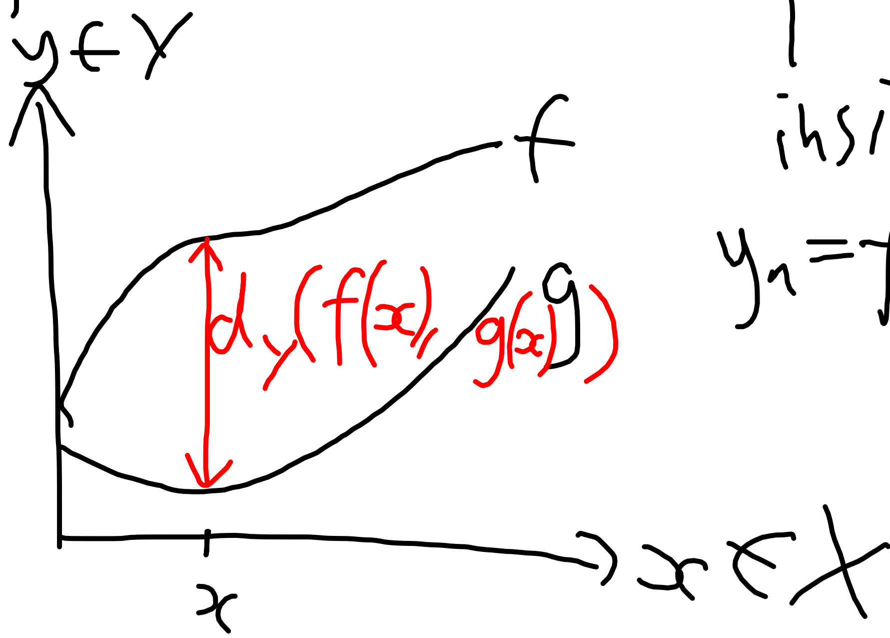
Proof idea:

$f_n \in B(X, Y)$.

$$f^*(x) = \lim_{n\to\infty} f_n(x).$$

inside Y

$y_n = f_n(x)$.



Fixed Points A.2.8

Def A function f is a *self-map* if $f: X \to X$.

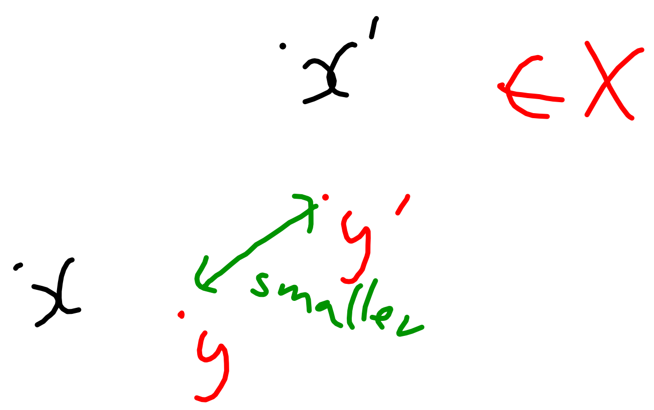
Def Let $f: X \to X$. A point $x^* \in X$ is a *fixed point* if $x^* = f(x^*)$.

Def Metric spaces:
(X, d_X), (Y, d_Y).

We say that $f: X \to Y$ is <u>Lipschitz continuous of degree "a"</u> if for every $x, x' \in X$,

$$d_Y(f(x), f(x')) \leq a\, d_X(x, x').$$

Def Let (X, d) be a metric space. A self-map $f: X \to X$ is a **contraction** if it is Lipschitz continuous of degree $a < 1$.



Banach's fixed point theorem

Let (X, d) be a complete metric space. If $f : X \to X$ is a contraction of degree a, then:

(i) f has a fixed point x^*.

(ii) Given any $x_0 \in X$, the sequence $x_{n+1} = f(x_n)$ converges to x^*.

(iii) $d(x_n, x^*) \leq \dfrac{a^n}{1-a} d(x_0, x_1)$.

<u>Proof</u>

<u>Uniqueness:</u> Suppose x^* and x^{**} are distinct fixed points.

$$\Rightarrow d(f(x^*), f(x^{**})) = d(x^*, x^{**}).$$

Because f is a contraction,

$$d(f(x^*), f(x^{**})) \leq a\, d(x^*, x^{**}) < d(x^*, x^{**}).$$

Contradiction.

Existence and convergence.

We will show x_n is Cauchy.

Formula 1: $d(x_n, x_{n+m})$
$$\leq a^n d(x_0, x_m).$$

Formula 2: $d(x_0, x_m) \leq \frac{1}{1-a} d(x_0, x_1).$

Proof: triangle inequality

$$d(x_0, x_m) \leq d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{m-1}, x_m)$$

$$\leq d(x_0, x_1) + d(x_1, x_2) + \cdots$$

infinite series

$$\leq d(x_0, x_1) + a\,d(x_0, x_1) + a^2 d(x_0, x_1) + \cdots$$

formula 1

$$= d(x_0, x_1)\left[1 + a + a^2 + \cdots\right]$$

$$= d(x_0, x_1)\frac{1}{1-a}$$

geometric series

Formula 3:

$$d(x_n, x_{n+m}) \leq \frac{a^n}{1-a} d(x_0, x_1).$$

for all n, m.

m not on RHS!

<u>Formula 4:</u>

$$d(x_n, x_m) \leq \frac{a^N}{1-a} d(x_0, x_1)$$

for all $n, m > N$.

$\Rightarrow x_n$ is a Cauchy sequence.

Since (X, d) is complete,

x_n is convergent.

Let $x^* = \lim_{n \to \infty} x_n$.

Since f is continuous

$$f(x_n) \to f(x^*).$$

Notice that $y_n = f(x_n)$ is a subsequence of x_n.

$(y_n = x_{n+1})$

Therefore $y_n \to x^*$

and $y_n \to f(x^*)$.

Therefore $x^* = f(x^*)$.

So x^* is a fixed point.

Approximation bound

$$d(x_n, x^*)$$

since $d(x_n, \cdot)$ is continuous

$$= \lim_{m \to \infty} d(x_n, x_m)$$

$$\leq \frac{a^n}{1-a} d(x_0, x_1)$$

Formula 4

$\square$