Def Let A be a set in the metric space (X, d).
The closure of A is

$$cl(A) = \overline{A}$$

$$= \{x^* \in X : \text{there is a sequence } x_n \in A \text{ with } x_n \to x^*\}.$$

e.g. $(0,1)$ is not closed in $(\mathbb{R}, d_2)$.

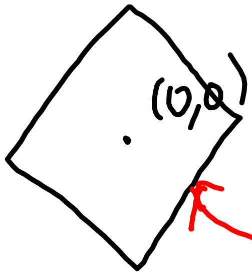
$$cl((0,1)) = [0,1].$$

Open Sets (A.2.5)

Def Let (X, d) be a metric space. Then open ball centred at $x \in X$ with radius $r > 0$ is $N_r(x) = \{y \in X : d(x,y) < r\}$

neighbourhood

e.g. in $(\mathbb{R}^2, d_1)$, $N_1((0,0))$



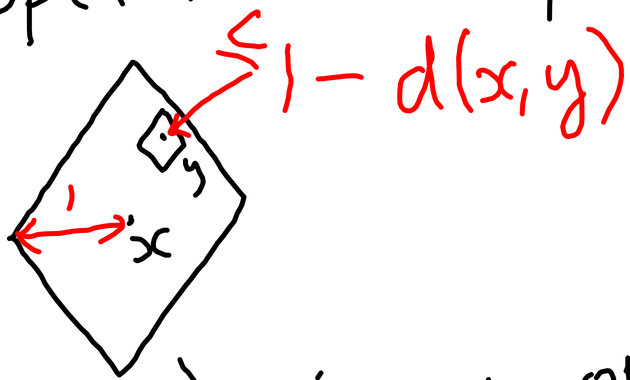
$N_1((0,0))$, boundary excluded

<u>Def</u> Suppose A is a subset of a metric space (X, d).
We say $x \in A$ is an <u>interior point</u> if there is an open ball $N_r(x)$ such that $N_r(x) \subseteq A$. The set of all interior points of A is called the <u>interior</u> of A, written $int(A)$. We say A is an open set if $A = int(A)$. If A is an open set and $x \in A$, then we say A is an <u>open</u>

<u>neighbourhood</u> of x.

<u>examples:</u>

* open balls are open sets



* $(0,1)$ is an open set $(\mathbb{R}, d_2)$
* If (X,d) is a metric space, then $\emptyset$ and X are open sets.
* $[0,1]$ is an open set in $([0,1], d_2)$, but not in

$(\mathbb{R}, d_2)$.

<u>Theorem</u> Let A be a subset of a metric space (X, d).

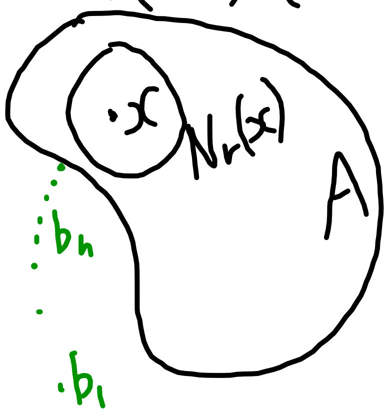
A is open $\iff A \cap \partial A = \emptyset$.

iff

A contains none of its boundary

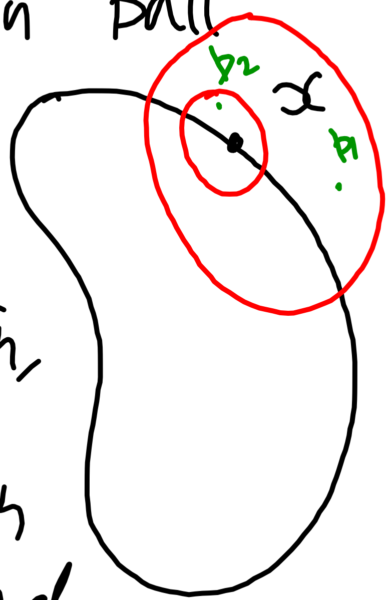
<u>Proof</u>

<u>$\Longrightarrow$</u>: Suppose A is open. Consider any point $x \in A$. Since A is open, there is some ball $N_r(x) \subseteq A$. There is no sequence $b_n \to x$ with $b_n \notin A$.



Therefore, $x \notin \partial A$.

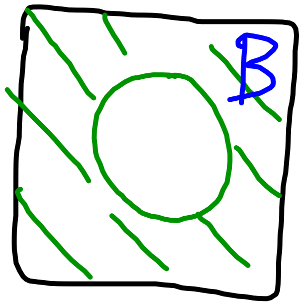
$\underline{\Leftarrow}$: Suppose A is not open.
This means there is some $x \in A$ such that every open ball $N_r(x) \not\subseteq A$. Let $r_n = \frac{1}{n}$. For each r_n, there is some point b_n s.t. $d(b_n, x) < r_n$ and $b_n \notin A$. Notice that $b_n \to x$.
We conclude $x \in \partial A$.
Therefore $x \in A \cap \partial A$.
So $A \cap \partial A \neq \emptyset$. $\square$



examples of open <u>and</u> closed sets:

* $\emptyset$ and X are both open and closed in (X,d).
* $[0,1]$ is open & closed in $([0,1] \cup [2,3], d_2)$.

<u>Theorem</u> Let A be a subset of a metric space (X, d).
A is open $\Longleftrightarrow$ $X \setminus A$ is closed.



Proof

Trick: $\partial A = \partial(X \setminus A)$.

look at the def of ∂A.

$\Rightarrow$: If A is open, it contains none of ∂A.

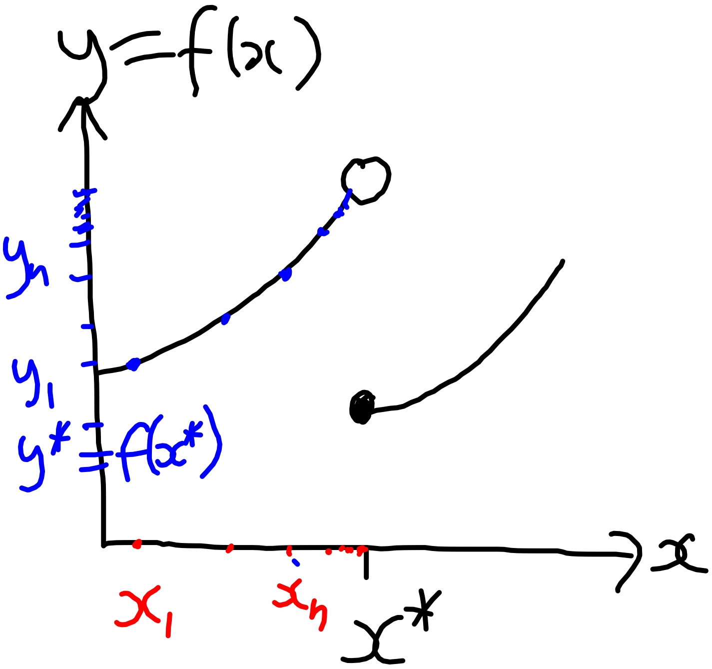
So $X \setminus A$ contains all of ∂A.

$\Rightarrow$ $X \setminus A$ contains all of $\partial(X \setminus A)$.

$\Leftarrow$: similar. $\square$

Continuity (A.2.6)

Def Consider two metric spaces (X, d_X) and (Y, d_Y).
We say $f: X \to Y$ is <u>continuous</u> at $x^* \in X$ if for every sequence $x_n \to x^*$, the sequence $y_n = f(x_n) \to y^* = f(x^*)$.



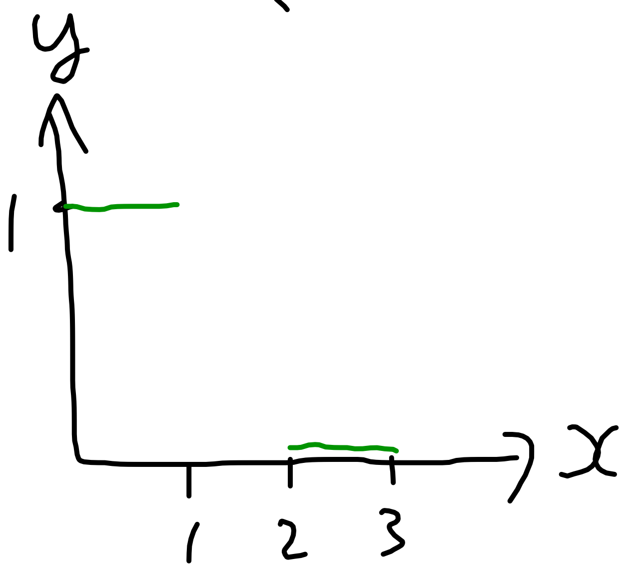
f is discontinuous because

$$y_n \not\to y^*$$

$$f(x_n) \not\to f(x^*).$$

even though $x_n \to x^*$.

* $f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \in [2,3] \end{cases}$



* If $f: X \to Y$ and (X, d_X) involves the discrete metric, then f is continuous.

Notation: if $f: X \to Y$

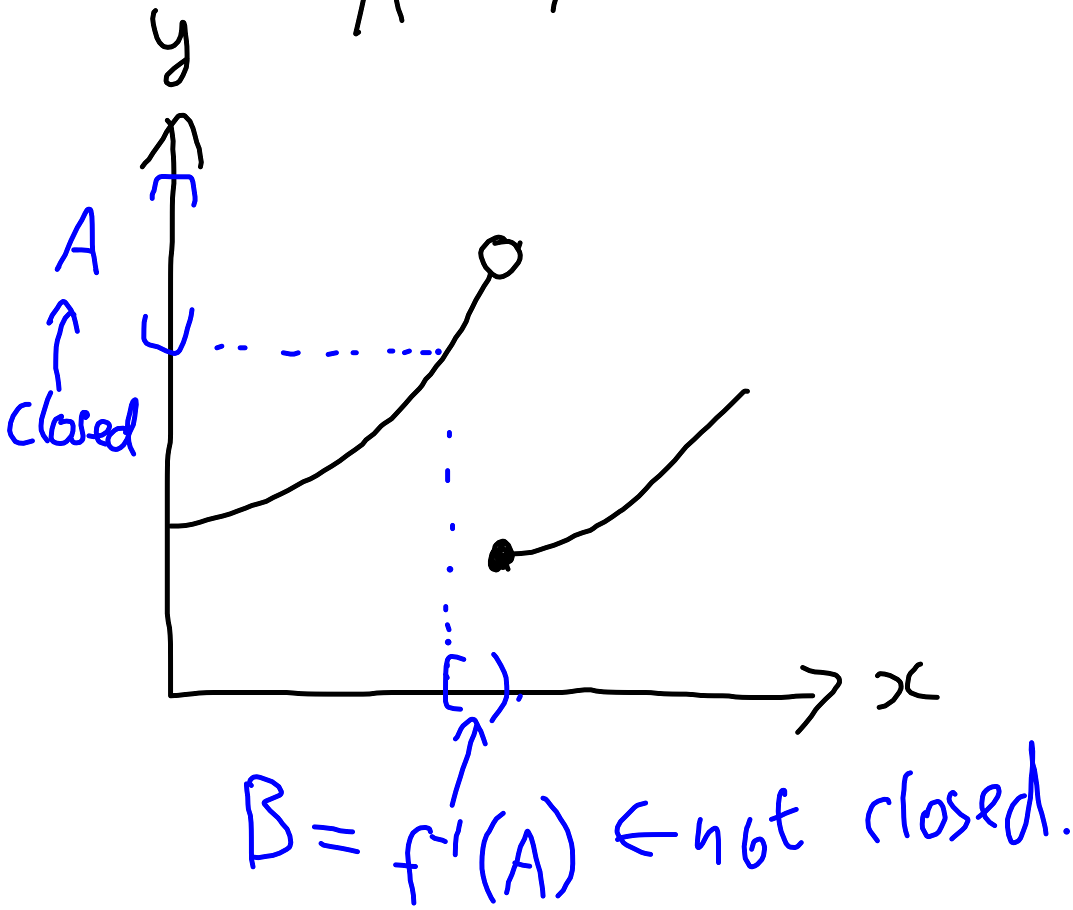
$$f(A) = \{ f(a) : a \in A \}$$

$$f^{-1}(B) = \{ x \in X : f(x) \in B \}.$$

<u>Theorem</u> Let $f: X \to Y$ be a function, where (X, d_X) and (Y, d_Y) are metric spaces.

(i) f is continuous $\iff$ $f^{-1}(A)$ is an open set for all open sets $A \subseteq Y$.

(ii) f is continuous
$\iff f^{-1}(A)$ is closed set
for all closed sets
$A \subseteq Y$.



Proof

(i) and (ii) are equivalent because A is open $\Leftrightarrow$ $X \setminus A$ is closed.

Specifically, if B is open, then $A = X \setminus B$ is closed.

So if $f^{-1}(A)$ is closed, then $f^{-1}(B) = X \setminus f^{-1}(A)$ is open.

$\Rightarrow$ using closed sets:

Suppose f is continuous.
Pick any closed set A in (Y, d_Y).
Want to prove: $B = f^{-1}(A)$ is closed.
Let $x_n \in B$ be any convergent sequence, and let $x^* = \lim_{n\to\infty} x_n$.
Want to prove $x^* \in B$.
By continuity, the sequence

$$y_n = f(x_n) \to y^* = f(x^*).$$

Since $y_n \in A$ and A is closed, $y^* \in A$. Therefore $f(x^*) \in A$

and hence $x^* \in B$.

$\Leftarrow$ open sets:

Suppose if $A \subseteq Y$ is open, then $f^{-1}(A)$ is open

~~it to show $f^{-1}(A) \subseteq X$ is~~

~~24~~. Let $x_n \to x^*$ pre-image of A.

be a sequence in (X, d).

Let $y_n = f(x_n)$, and $y^* = f(x^*)$.

Want to prove $y_n \to y^*$.

Pick any $r > 0$.

Since $A = N_r(y^*)$ is open set. So $f^{-1}(A)$ is an open set, and $f^{-1}(A)$ contains x^*. Since $x_n \to x^*$, there some N such that for all $n > N$, we have $x_n \in f^{-1}(A)$

will fix the rest of the proof in the notes.

Completeness A.2.7

$((0,1], d_2) = (X, d)$

$x_n = \frac{1}{n}$ wants to converge, to 0, but $0 \notin X$.