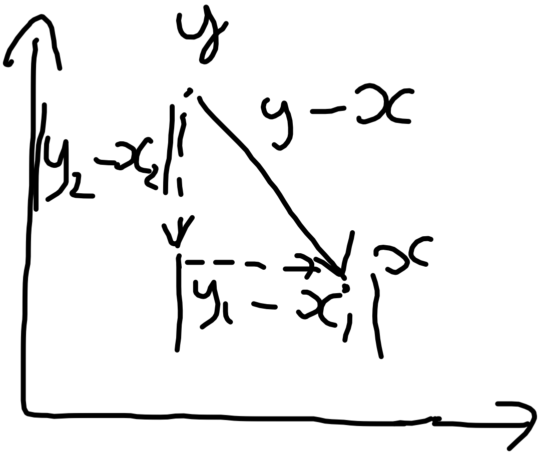
$$x, y \in \mathbb{R}^N$$

$$d(x, y) = \sqrt{\sum_i (x_i - y_i)^2}$$

Euclidean distance



Def (X, d) is a **metric space** with point set X and distance metric $d: X \times X \to \mathbb{R}_+$ if

(i) $d(x,y)=0 \iff x=y$

(ii) $d(x,y)=d(y,x)$ for all $x,y \in X$

(iii) $d(x,z) \leq d(x,y)+d(y,z)$ for all $x,y,z \in X$

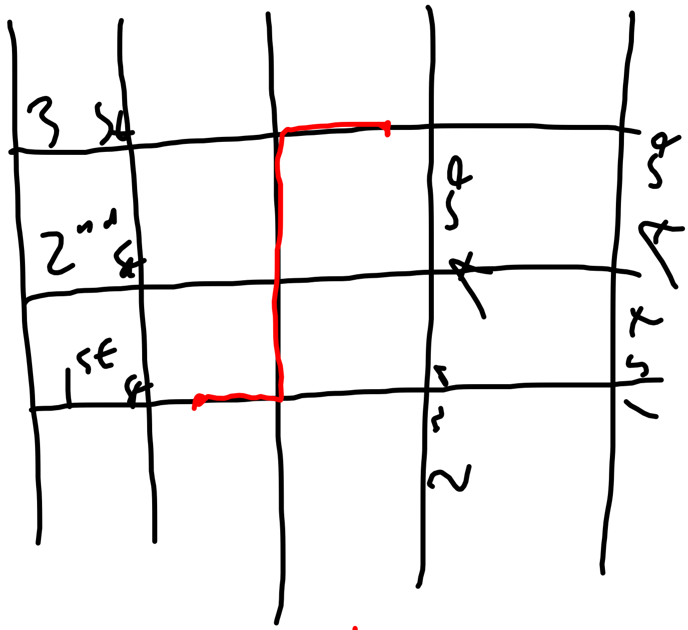
triangle inequality

Examples of metric spaces:

* $(\mathbb{R}^N, d_2)$ ← Euclidean metric

* $(\mathbb{R}^n, d_1)$ where

$$d_1(x,y) = \sum_{i=1}^{N} |x_i - y_i|$$



Manhattan metric

* Given (X,d), if $Y \subseteq X$ then (Y, d_Y) is a metric space, where
  $$d_Y(x,y) = d(x,y) \text{ for all } x,y \in Y$$

cheat: (Y, d)

* (X, d), where

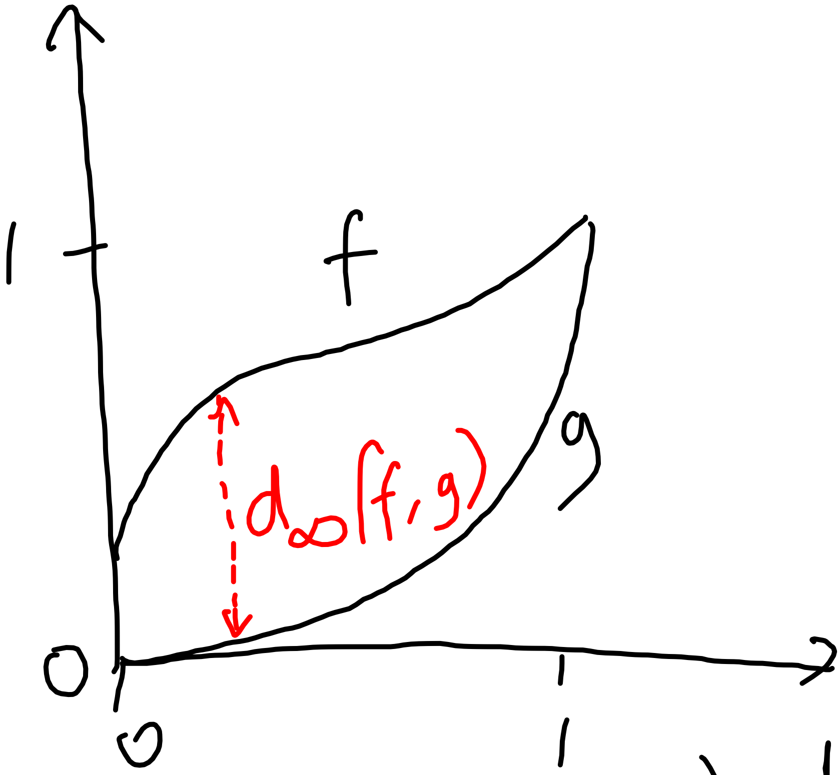
$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

discrete metric

* (X, d_∞) where

$$X = \{ f : [0,1] \to [0,1] \}$$

$$d_\infty(f,g) = \sup_{a \in [0,1]} |f(a) - g(a)|$$

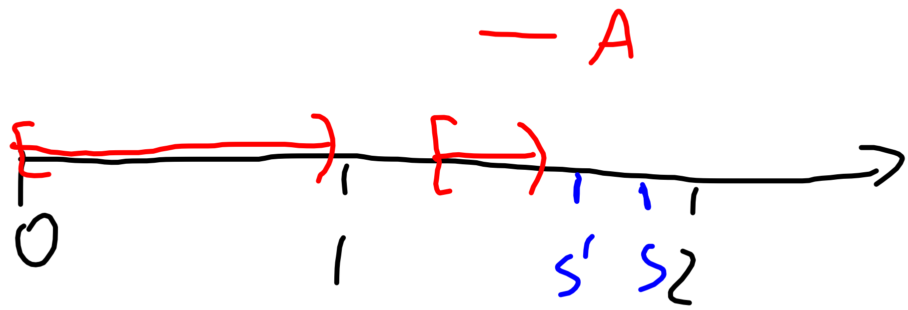


Problem: $\max [0,1)$ does not exist!

$\{a : 0 \leq a < 1\}$

sup (supremum)

$\sup A = \min \{ s \in \mathbb{R} : a \leq s$ for all $a \in A \}$



$* \ \ell_\infty =$ bounded sequences

$= \{ x_n :$ each $x_n \in \mathbb{R}$ and there is $r>0$ s.t. $|x_n| < r$ for all $n \}$

$n \in \mathbb{N}$

5, 10, 8, 3, . . .

e.g. $r = 20$

$$d_\infty(x_n, y_n) = \sup_{n \in \mathbb{N}} |x_n - y_n|$$

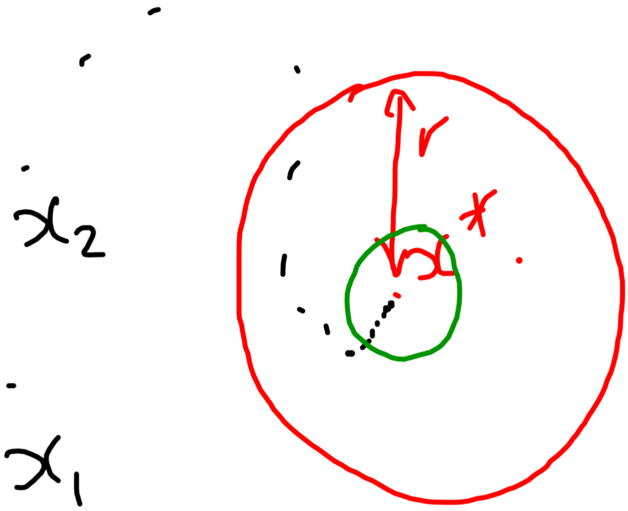
Seq's & Convergence A.2.2

Def A sequence in a set X is a function with domain $\mathbb{N}$ and co-domain X. Notation: "x_n is a sequence in X"

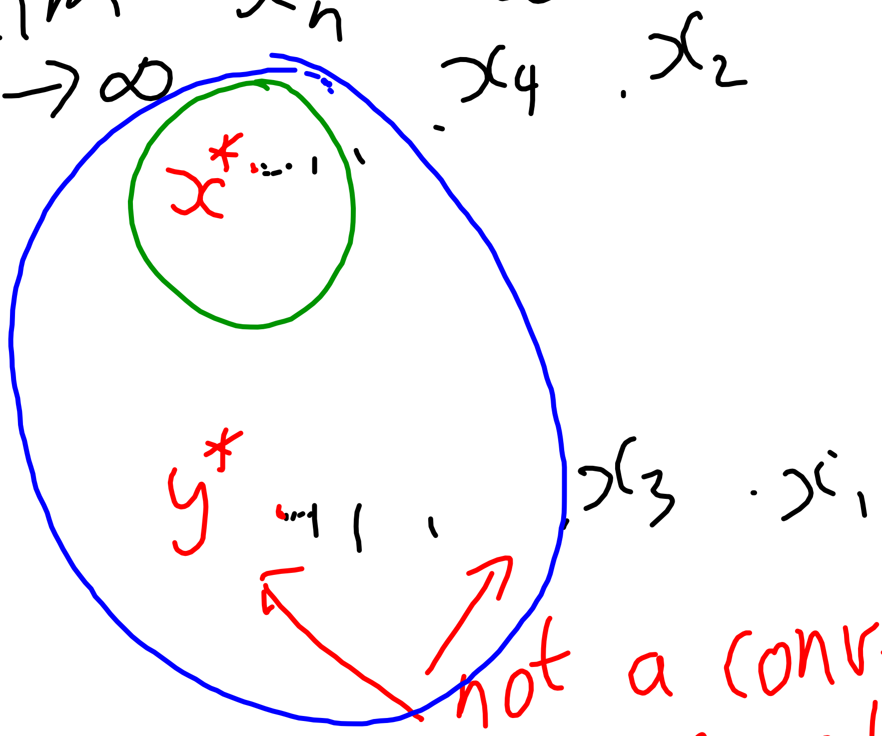
$x_1, x_2, x_3, \ldots$

$\{x_n\}$ $\{x_n\}_{n=0}^{\infty}$

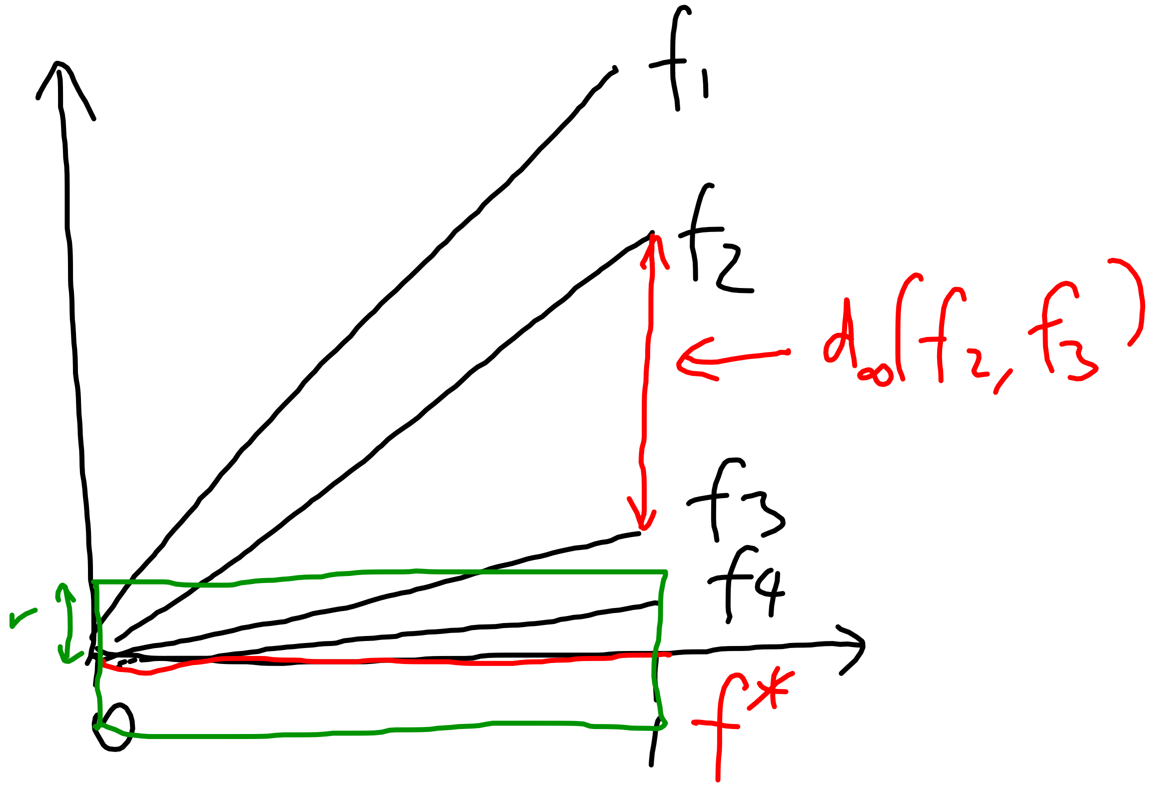
<u>Def</u> Let x_n be a sequence in a metric space (X, d). We say that x_n <u>converges</u> to x^* $(x_n \to x^*)$ if for every $r > 0$, there is some $N > 0$ such that $d(x_n, x^*) < r$ for all $n > N$.



$$\lim_{n \to \infty} x_n = x^*$$



not a convergent sequence!



$f_n \to f^*$ using d_∞?

$d_\infty(f_n, f^*) = f_n(1)$

Theorem A sequence x_n in a metric space (X, d) can converge to at most one point.

Proof For the sake of contradiction, suppose $x_n \to x^*$ and $x_n \to y^*$ and $x^* \neq y^*$.
Let $r = \frac{1}{2} d(x^*, y^*)$.
Since $x_n \to x^*$ and $x_n \to y^*$, there is some N s.t.
$d(x_n, x^*) < r$ and $d(x_n, y^*) < r$

for all $n \geq N$.

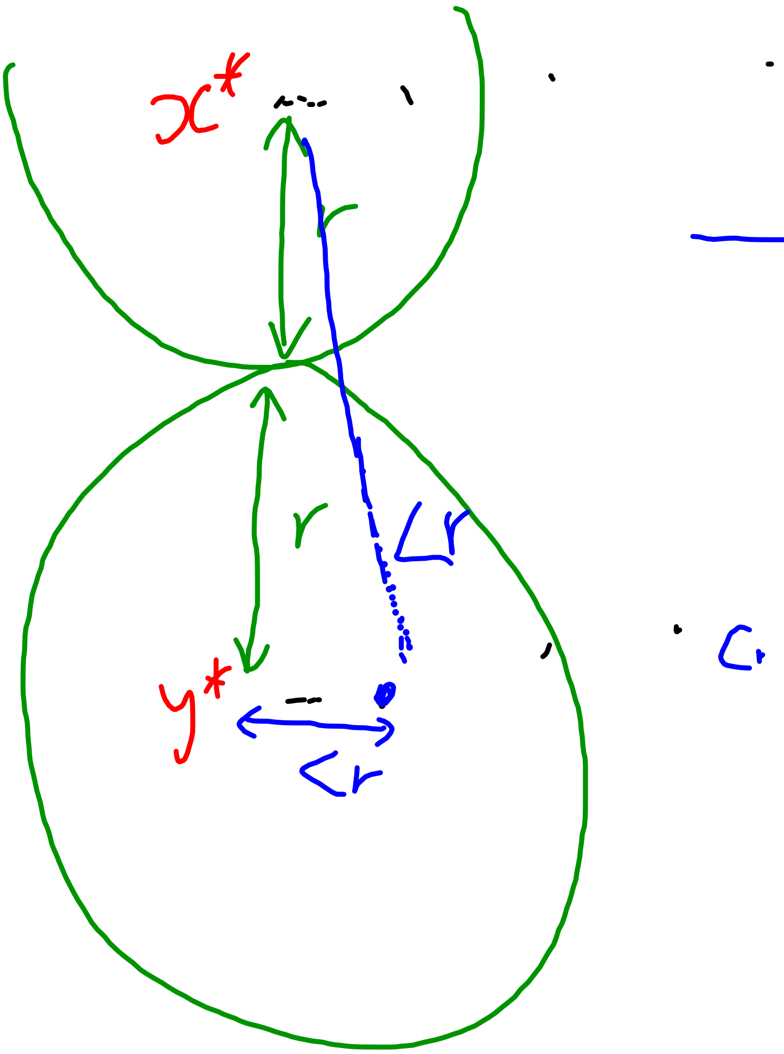
We conclude

$$d(x^*, y^*) = r + r$$
$$> d(x_N, x^*) + d(x_N, y^*)$$

But the triangle inequality says

$$d(x^*, y^*) \leq d(x^*, x_N) + d(x_N, y^*).$$

Contradiction! $\square$



— short-cut (forbidden!)

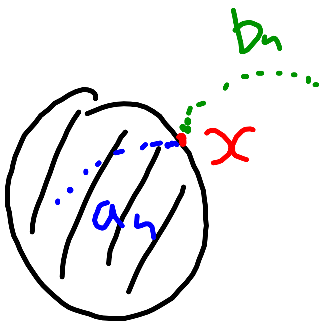
Def We say y_n is a subsequence of x_n if there is some increasing sequence $k_n \in \mathbb{N}$ such that

$$y_n = x_{k_n} \text{ for all } n.$$

eg $k_n = 1, 3, 5, 7, \dots$

<u>Theorem</u> If $x_n \to x^*$ and y_n is a subsequence of x_n, then $y_n \to x^*$.

Boundaries A.2.3



Def Let A be any subset of a metric space (X, d). A point $x \in X$ is a **boundary point** of A if

(i) there is a sequence $a_n \in A$

with $a_n \to x$, and

(ii) there is a sequence $b_n \cancel{\notin A} \in X \setminus A$ with $b_n \to x$.

The <u>boundary</u> of A, denoted ∂A, is the set of boundary points of A.

Boundary of $[0,1]$ in $(\mathbb{R}, d_2)$

$= \{0,1\}$

Boundary of $[0,1]$ in $(\mathbb{R}_+, d_2)$

$= \{1\}$

Boundary of $(0,1)$ in $(\mathbb{R}_+, d_2)$

$= \{0,1\}$

$a_n = 0$. $a_n \to 0$.

$a_n \in X \setminus A$

Boundary of $(0,1)$ in $([0,1], d_2)$

$= \{0,1\}$.

Closed sets A.2.4

Def A is a **closed** subset of (X,d) if there is no sequence $a_n \to a^*$ such that $a_n \in A$ for all n but $a^* \notin A$.

e.g. $[0,1]$ is a closed set in $(\mathbb{R}, d_2)$

$(0,1]$ is a closed set in $(\mathbb{R}_{++}, d_2)$.

but <u>not</u> in $(\mathbb{R}_{+}, d_2)$.

eg: $a_n = \frac{1}{n}$

Theorem Consider a set A in a metric space (X,d). Then A is closed if and only if $\partial A \subseteq A$.