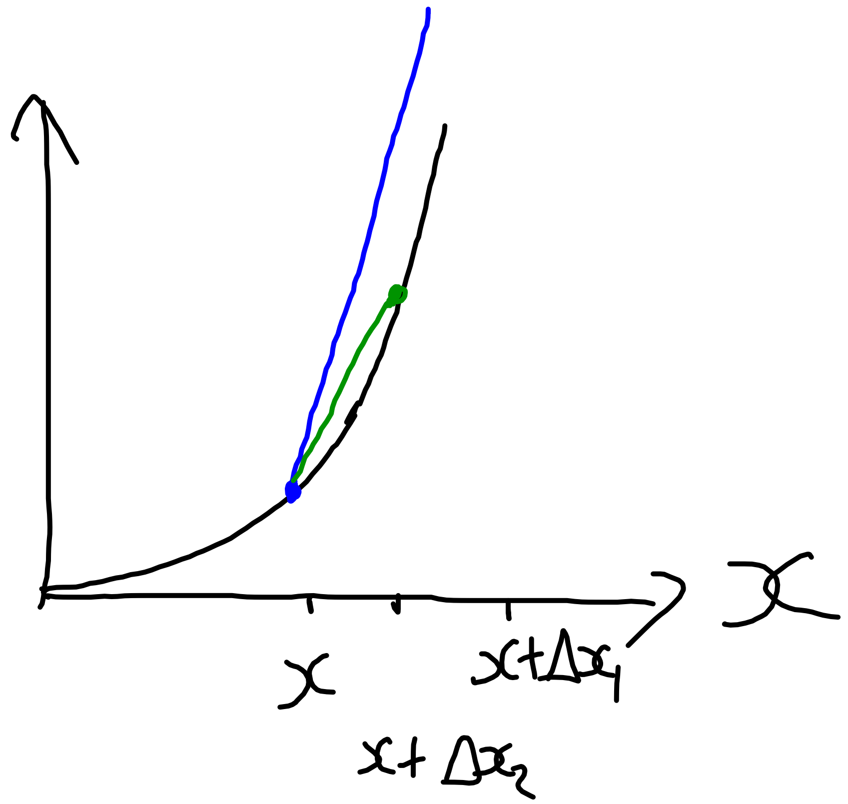
High school calculus

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

pick any sequence $\Delta x_n \in \mathbb{R}\setminus\{0\}$
s.t. $\Delta x_n \to 0$.
Undefined UNLESS you
get the same answer
for every sequence.



Derivative is
* a slope (a single number)
* limit of slopes

Def Consider two functions $f, g : \mathbb{R}^n \to \mathbb{R}^m$.
We say g is a first-order approximation of f at x^* if

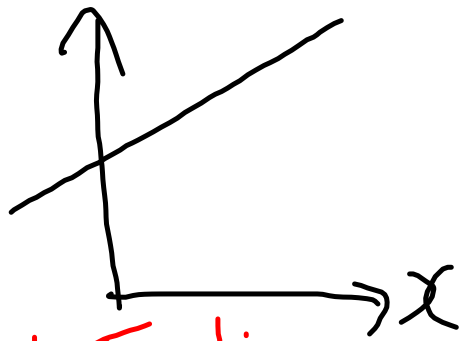
$$\lim_{\Delta x \to 0} \frac{f(x^* + \Delta x) - g(x^* + \Delta x)}{\|\Delta x\|} = 0$$

$\sqrt{x_1^2 + x_2^2 + \cdots x_n^2}$

<u>Def</u> A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is a <u>linear function</u> if

* $f(x+x') = f(x) + f(x')$ for all $x, x' \in \mathbb{R}^n$
* $f(tx) = t f(x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Note: $f(0)=0$.



NOT linear

~~affine~~ affine.

If $m=1$, then if f is linear, it can written as

$$f(x) = d \cdot x$$

for some $d \in \mathbb{R}^n$.

<u>Def</u> The function $f: \mathbb{R}^n \to \mathbb{R}^m$ is <u>differentiable</u> at x^* if there is some linear function $g: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$x \longmapsto g(x) + f(x^*) - g(x^*)$$

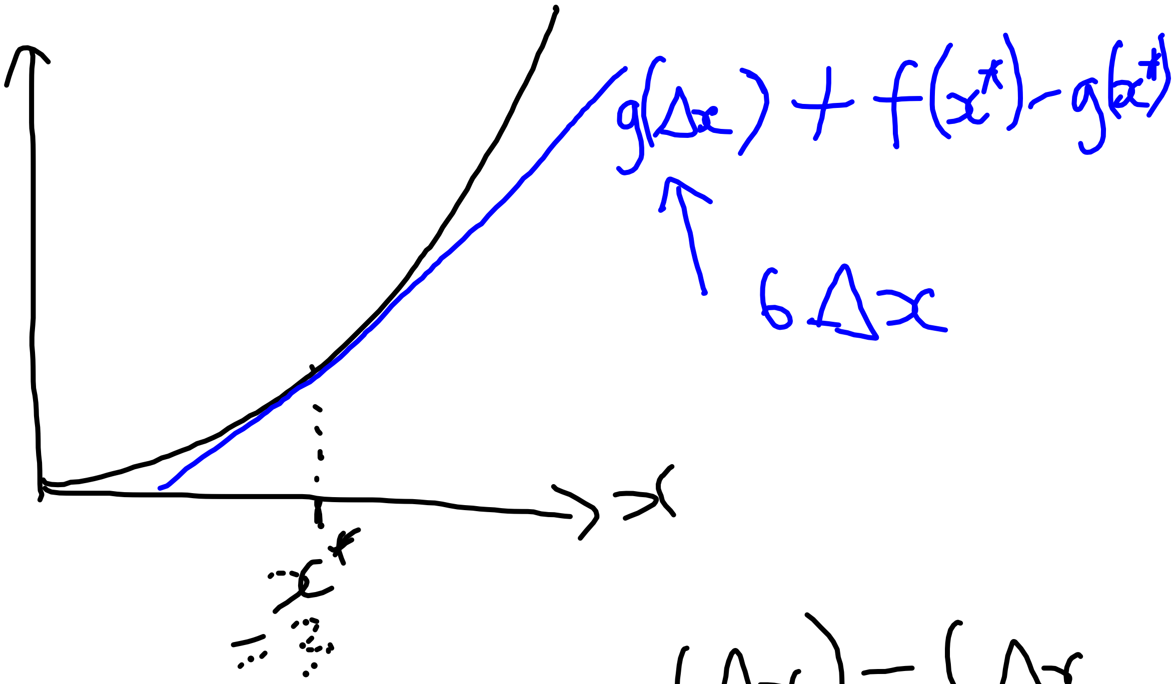
is a first order approximation of f at x^*.

Specifically if $m=1$, then f is differentiable at x^* if there is some $d \in \mathbb{R}^n$ such that

$$\lim_{\Delta x \to 0} \frac{f(x^* + \Delta x) - [d \cdot \Delta x + f(x^*)]}{\|\Delta x\|} = 0.$$

E.g. $f(x) = x^2$.

Derivative at $x^* = 3$?



Derivative is $g(\Delta x) = 6\Delta x$.

$$u(x,y) = \log(xy)$$

Derivative of u at $(x,y) = (1,2)$

$$\underbrace{v(x,y)}_{\text{derivative at } (1,2) \text{ according to def.}} = \underbrace{\left(1, \tfrac{1}{2}\right)}_{\text{short-cut representation}} \cdot (x, y).$$

$Du(x,y)$

d

Non-differentiable function:

$$f(x,y) = \begin{cases} 0 & \text{if } x=0 \text{ or } y=0 \\ 1 & \text{otherwise} \end{cases}$$

This function is not diff'able at $(0,0)$.

Note: $\frac{\partial f}{\partial x}(0,0) = 0$

$\frac{\partial f}{\partial y}(0,0) = 0.$

Consider $\Delta x_n = \frac{1}{n}(1,1)$.

$\Delta x_n \to (0,0)$.

$$\lim_{n\to\infty} \frac{f(\overset{(0,0)}{x^*} + \Delta x_n) - [(0,0)\cdot \Delta x_n + f(x^*)]}{\|\Delta x_n\|}$$

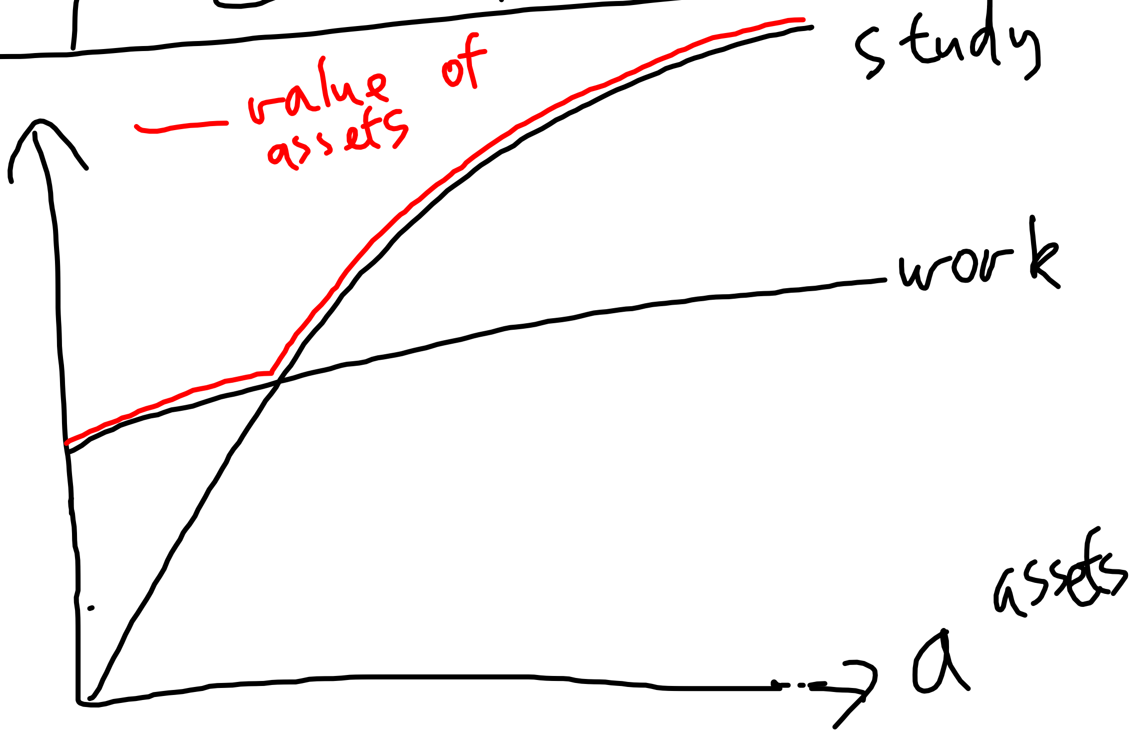
$$= \lim_{n\to\infty} \frac{1 - [0+0]}{\left\|\frac{1}{n}(1,1)\right\|} = \frac{1}{\frac{1}{n}\underbrace{\|(1,1)\|}_{\sqrt{2}}}$$

$$= \lim_{n\to\infty} \frac{n}{\sqrt{2}} \to \infty \neq 0.$$

So f is not differentiable.

Theorem If $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$, then f is continuous at x.

F.4 Envelope Theorem



<u>Def</u> Let $X \subseteq \mathbb{R}^n$.

Consider the functions $f, g : X \to \mathbb{R}$. We say that g is a <u>differentiable lower support function</u> of f at $\bar{x} \in X$ if

$\geq$ upper

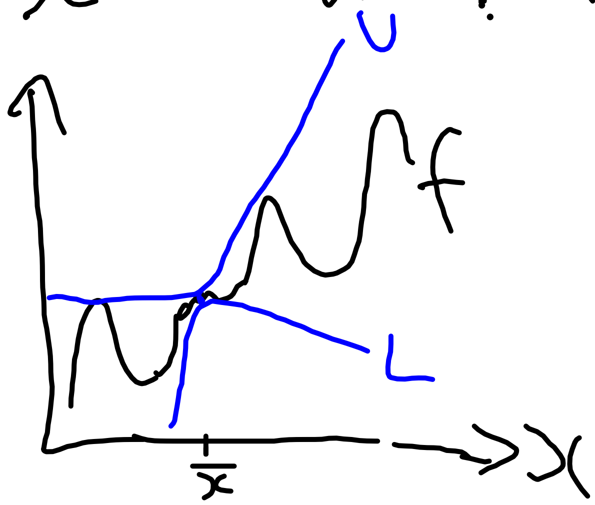
(i) $f(\bar{x}) = g(\bar{x})$

(ii) $g(x) \leq f(x)$ for all $x \in X$

(iii) g is differentiable at $\bar{x}$.

Lemma (Differentiable Sandwich Lemma)

If F has differentiable lower and upper support functions L and U at $\bar{x}$, then F is differentiable at $\bar{x}$ and $F'(\bar{x}) = L'(\bar{x}) = U'(\bar{x})$.



<u>Proof</u> Let $d(x) = U(x) - L(x)$.

Note: $d(x) \geq 0$ for all x,

$d(\bar{x}) = 0$,

d is differentiable at $\bar{x}$.

FOC (since d is minimised at $\bar{x}$):

$$d'(\bar{x}) = 0.$$

$$\Rightarrow U'(\bar{x}) - L'(\bar{x}) = 0$$

$$\Rightarrow U'(\bar{x}) = L'(\bar{x}).$$

Let $m = U'(\bar{x}) = L'(\bar{x})$.

$$\frac{L(\bar{x}+\Delta x) - [F(\bar{x}) + m \cdot \Delta x]}{\|\Delta x\|} \longrightarrow 0$$

$$\leq \frac{F(\bar{x}+\Delta x) - [F(\bar{x}) + m \cdot \Delta x]}{\|\Delta x\|}$$

$$\leq \frac{U(\bar{x}+\Delta x) - [F(\bar{x}) + m \cdot \Delta x]}{\|\Delta x\|} \longrightarrow 0$$

$\Rightarrow$ middle $\to 0$.

$\Rightarrow$ F is differentiable at $\bar{x}$ with derivative m. $\square$

<u>Theorem F.4</u> (Benveniste-Scheinkman 1979)

Let $X \subseteq \mathbb{R}^n$ and consider

$$F(x) = \max_{y \in Y} g(x, y)$$

with $F: X \to \mathbb{R}$. Let $y(x)$ be an optimal policy.

If F is a concave function and each $g(\cdot, y)$ is differentiable for each $y \in Y$, then F is differentiable at $x \in X$ with

$$F'(\bar{x}) = \left[\frac{\partial g(x, y(\bar{x}))}{\partial x}\right]_{x=\bar{x}}$$

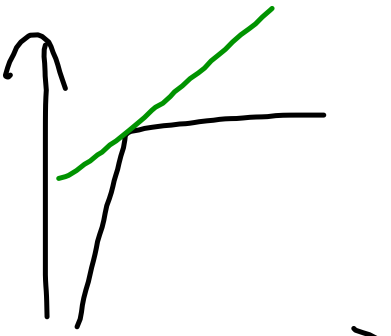
Proof:

Lower support function:

$$L(x) = g(x, y(\bar{x})).$$

last word

Upper support function:



support hyperplane theorem

$U(x)$

Then the differentiable sandwich Lemma implies F is differentiable at $\bar{x}$, and

$$F'(\bar{x}) = L'(\bar{x})$$
$$= \text{RHS of formula. } \square$$