# Practice Questions

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**Notes for Microeconomics 1 students.** My part of the class and degree exams have an identical format and marking scheme, approximately as follows (not counting the bonus questions). You will be rated as no/almost/ok/good/excellent (i.e. 0 to 4) on the following learning outcomes:

- Formulating a model. Excellence here means the absence of important mistakes.
- Breadth of technique: Walras law, dynamic programming, the envelope theorem, convex analysis, the first welfare theorem, etc. Excellence here usually means applying four techniques.
- Depth of technique: using a technique in an unusual way, combining several techniques to deduce something, or a clever piece of logic. For example, proving that there is at most one equilibrium in a particular model by combining the first welfare theorem with symmetry of all households. Obviously, depth requires at least some breadth, so this is correlated with the breadth learning outcome. Excellence here means that the assumptions, conclusions, and the logic from one to the other are clearly expressed.

There is no precise system for determining marks, but a linear regression reveals that marks will usually be close to $45.5 + 2.7m_1 + 3.8m_2 + 4.4m_3$, where $m_i$ is the mark on learning outcome $i$ on the $0 - 4$ scale. This formula is less accurate at both extremes – all marks between 0 and 100 are possible.

Note that exams have become longer and more difficult in recent years to give multiple opportunities to show the third learning outcome. Thus, it has become easier to get a high mark.

**Notes for Advanced Mathematical Economics students.** You will be rated as no/almost/ok/good/excellent (i.e. 0 to 4) on the following learning outcomes:

- Formulating a model. Excellence here means the absence of important mistakes.
- Applying theorems to models. Excellence here usually means applying three techniques correctly.

- Proving mathematical theorems. This is by far the hardest learning outcome. Excellence here means proving something difficult (most of the questions in Part B qualify) with clear reasoning.

Earning first class honours (70+) requires demonstrating all three learning outcomes. There is no precise system for determining marks, but a linear regression reveals that marks will usually be close to $49 + 2.1m_1 + 2.5m_2 + 5.6m_3$, where $m_i$ is the mark on learning outcome $i$ on the $0 - 4$ scale. This formula is less accurate at both extremes – all marks between 0 and 100 are possible. I strongly recommend that students attempt Part A first.

Questions 20, 21, 24, and 27 are specifically written for Advanced Mathematical Economics. All other questions are from Microeconomics 1, which covers different material. Many questions are based on material that was taught, but only in passing. These questions would be examinable as more advanced questions (and hence would attract a bigger reward if correctly answered.) These questions are *not* examinable, as they are based on ideas that were not covered in lectures at all:

1 (v), (vi), (vii).

2 all examinable.

3 (ii), (vii), (viii).

4 (iv), (vi), (vii).

5 (vi).

6 (ii), (vi), (vii).

7 (ii), (vi).

8 (ii), (iv).

9 (v).

10 (ii), (v), (vi).

11 (vi).

12 (iii), (v).

13 (ii), (vi), (vii).

14 (vii).

15 (ii), (v), (vi).

16 (iv), (vi).

17 (ii), (vi).

18 (ii), (vi), (vii).

19 (vi).

20 –

21 –

22 (iv), (v), (vi).

23 (ii), (v), (vi), (vii).

24 –

25 (ii), (iv), (v).

26 (iv), (v), (vi).

27 –

28 (ii), (vi).

29 –

30 (ii), (iii), (iv), (v).

# Advice for Answering Exam Questions

**Generic Advice.**

- There is no need to add extra complications into the model. For example, if the question does not mention time, then there is no need to put multiple time periods into the model.

- If you can't figure out the answer, don't pretend you know it. It's better to explain what you are confused about – a well written statement of confusion can illustrate that you know the material very well, and give you a very good mark.

- Even if you misformulate your model, this shouldn't stop you from answering subsequent parts. But if the model then seems inconsistent with the question (e.g. the question asks "show real wages are higher" when in your model, this is not true) then please do not try to prove the impossible. Instead, please either explain why the question is inconsistent, or if you're not sure, explain why you are stuck and can't complete your argument.

- Students often incorrectly identify the envelope formula as a first-order condition. It's not. First-order conditions are about optimal choices. If you are differentiating with respect to prices, you are not doing a first-order condition, because in competitive markets, nobody can choose prices.

- Students often confuse value functions and objective functions. For example, students often (mistakenly) write that a firm's first-order conditions with respect to the number of workers involves differentiating the profit function (rather than the firm's objective function) with respect to its labour input. But the profit function is a function of prices, not quantities, so it makes no sense to differentiate it with respect to a quantity.

- You can introduce assumptions at any point in the paper. For example, if you discover in part (iv) that you need to assume that the production function is concave, then you can write that assumption in your answer to part (iv). You do not need to revise your answer to part (i).

- In proofs and calculations, please write with complete grammatical sentences, including punctuation.

- In proofs, be careful to distinguish between "there exists" and "for all".

- In proofs, be careful to distinguish between set membership ($\in$) and subsets ($\subseteq$).

**A Basic Checklist.** A mark below 50% means something important was missing from your model. For example, you might have had two different markets with the same price, or a firm buying something (like a wholesale good) without using it in production. Here is a check-list of important ingredients of every economic model:

- Any notation is fine, but you must define it.

- When writing down the agents' optimisation problems, you should always write the choice variables under the *max*.

- In competitive models, agents only choose quantities, not prices.

- Every market has one (and only one) price. For example, labour markets have only one price if all types of labour are equally valued (by buyers and sellers). On the other hand, if workers have preferences over their profession, or firms value some workers above others, then these are separate markets and have separate prices.

- Every cost should also have a corresponding benefit (and vice versa). There are exceptions to this rule (e.g. inelastic labour supply), but think carefully about this.

- Every market should have a market-clearing condition. Thus, there are always an equal number of prices and market-clearing equations. It also means you need to define notation for both supply and demand. (In the sample solutions, I typically write firm decisions in upper case, and household decisions in lower case.)

**Notation:** Notation for partial derivatives: there are many common (correct) ways to write partial derivatives, including

$$\frac{\partial}{\partial x} f(x,y) \tag{1}$$
$$\frac{\partial f(x,y)}{\partial x} \tag{2}$$
$$f_x(x,y) \tag{3}$$
$$f_1(x,y) \tag{4}$$
$$D_x f(x,y) \tag{5}$$
$$D_1 f(x,y) \tag{6}$$
$$\nabla_x f(x,y) \tag{7}$$
$$\nabla_1 f(x,y). \tag{8}$$

Writing

$$f'_x(x,y) \tag{9}$$

is *not* standard, so I suggest you avoid it. (It is unambiguous though, so it wouldn't lose you marks in my exams.)

The notation $f'(x,y)$ or $Df(x,y)$ or $\nabla f(x,y)$ does *not* represent a partial derivative, but rather the *total* derivative, i.e. the vector (or matrix) of partial derivatives of $f$. Please don't write this if you mean a partial derivative.

*Question* 1. Consider a pure-exchange economy in which all goods are produced from oil by home production over 2 time periods. Only oil is traded. There are two households and two oil deposit sites of size 1. The first site is owned by household A, and oil can be extracted from it at any rate over the 2 periods. The second site is owned by household B, but oil production is only possible in the second period. Both households have the same preferences, which are impatient discounted utility with the same per-period utility function which is strictly concave.

(i) Define an equilibrium in this economy.

(ii) Write down the egalitarian social planner's problem (i.e. assuming that the social planner puts equal weight on the households.) What allocation would she choose?

(iii) In equilibrium, how do oil prices change over time?

(iv) In equilibrium, which household is better off? Explain.

(v) Suppose there is a bubble, in the sense that in the last period, oil prices are too high and there is excess supply of oil in the last period. What would happen in the first period? (Hint: Walras' law.)

(vi) * Which assumptions above about the households' utility are relevant for Debreu's theorem about additively separable preferences? Which assumptions go beyond the conclusion of Debreu's theorem?

(vii) * What additional assumptions are needed to ensure existence of equilibria in this economy?

*Question* 2. The cashew tree is native to the Amazon forest in Brazil, its fruit is about the same size as an apple. The juice of the flesh of the fruit is popular in Brazil (along with açaí, acerola, guava, mango, papaya, and many others... but ignore those!) Each fruit also contains a single seed, which when toasted becomes the cashew nut which is popular all over the world.

(i) The firm chooses how many cashew fruits to grow (which requires labour), and then sells the juice and nuts. Assume that no work is required to extract the juice and nuts – only growing requires labour. Write down the firm's profit function.

(ii) Write down the firm's cost function. Hint: you will need two quantities in the state variable (as well as factor prices).

(iii) There are several identical households that supply labour and consume cashews and cashew juice and hold equal shares in the cashew firm. Write down a general equilibrium model of the economy.

(iv) Write down a utility function for the households consistent with the idea that households enjoy cashew nuts more than cashew juice. What can you say about equilibrium prices in this case?

(v) Does the firm have increasing marginal cost in both products?

(vi) Sketch a graph of the firm's marginal cost of producing cashew juice, holding fixed the number of cashew nuts being produced at 3.

*Question* 3. (Micro 1 class exam in December 2012) A farm produces food from labour. However, the farm does not have a distribution network, so it can not sell the food directly to the households. Rather, it must sell the food to a supermarket at a wholesale price, which then resells to households at a retail price. The supermarket buys food and labour, which it uses to resell the food. Some food might get wasted; more labour means less food gets wasted. All households are identical, and supply labour to both firms.

(i) Formulate an economy by writing down the households' and firms' value functions, and the market clearing conditions. Focus attention on symmetric equilibria, i.e. in which all households make the same decisions. (Hint: you might find it helpful to consider the wholesale food a completely separate good. Don't forget profits.)

(ii) Select a constraint which may be dropped by Walras' law.

(iii) Suggest how an endogenous variable may be eliminated, since inflation of all prices by an equal factor does not affect decisions.

(iv) Show that the supermarket's profit function is convex. (Hint, you may use the following theorem from class: Suppose $V$ is the upper envelope of convex functions, i.e. $V(a) = \max_b v(a, b)$ where $v(\cdot, b)$ is a convex function for each $b$. Then $V$ is convex.)

(v) Show that the supermarket responds to a wholesale price increase by buying less.

(vi) There have been protests recently that the (equilibrium) retail price is much higher than the wholesale price, which the households feel is grossly unfair. They propose introducing a profit tax of 50% to be redistributed equally among households, a price markup ceiling of 10%, and a minimum wage increase of 20%. Would this policy make the households better off (under standard assumptions, like increasing utility functions)?

(vii) * Prove that the supermarket's policy is continuous if its production function is strictly concave. You may assume that the supermarket only has space to accommodate a maximum number of workers and amount of food.

(viii) * To prove existence of equilibrium using Brouwer's fixed point theorem, it is important that the set of possible prices are compact. Explain why this is important, and how to accommodate this requirement.

*Question* 4. (Micro 1 class exam in December 2012) Sackman, Erickson, and Grant (1968) conducted an experiment on computer programmers, which they published in the Communications of the Association of Computing Machinery. They summarised their findings with the following poem:

> When a programmer is good,
> He is very, very good,
> But when he is bad,
> He is horrid.

Even though the programmers were quite experienced, there was very wide disparity in their abilities. They found the best programmer writes their code about 20 times more quickly than the worst programmer. They debug it 28 times more quickly, the final code runs about 10 times faster, and so on. Follow-up studies report similar disparities, and it has become conventional wisdom that the best computer programmers are about 10 times more productive than the median.

Suppose there is a mediocre and a brilliant computer programmer. Assume that one hour of work by the brilliant programmer is a perfect substitute for ten hours of work by the mediocre programmer. The households are otherwise identical and hold equal shares in the firm.

(i) Write down a model of this economy, and define a general equilibrium for it.

(ii) Show that in every equilibrium in which both programmers are hired, the brilliant programmer's wage is ten times higher than the mediocre programmer's wage.

(iii) Show that in every equilibrium, the brilliant programmer is better off than the mediocre programmer.

(iv) Depending on the preferences of the households, the brilliant programmer might work longer or shorter hours. Draw the indifference curves in a way that indicates the brilliant programmer working *less* than the mediocre programmer.

(v) Some people think that the problem is that mediocre programmers are lazy, and they just need some extra incentives to work hard. In the context of your model, would giving the programmers stock options, 100% bonus pay upon project completion and hiring a masseuse and celebrity chef make everyone better off?

(vi) The mediocre programmer has another more Machiavellian proposal for increasing productivity. He proposes asking the government to issue a large lump-sum tax on the brilliant programmer, which will force her to work long hours to repay her (government-imposed) debt. The mediocre programmer further proposes the he receive the taxes. Would this proposal work?

(vii) * Discuss the problems with proving existence in this economy.

*Question* 5. (Micro 1 degree exam in May 2013) We eat about 300 billion apples every year, but most of these apples can not be eaten directly from the tree. The problem is that apples only ripen in Autumn, and apples consumed at other times must be stored. On the other hand, lettuce may be grown in all seasons, so it is never necessary to store it. Henceforth, assume it is non-storable.

Suppose there are just two seasons (Autumn and Spring) and two foods (lettuces and apples). Farmers are endowed with apples in Autumn, and lettuce in equal quantities in both Autumn and Spring. There is a storage firm (owned by the farmers) that can refrigerate apples until the Spring. The storage technology does not require any labour or other resources to operate. However, as they store more fruit, they become less effective and an increasing fraction of apples go bad.

(i) Define a general equilibrium in this setting, focusing attention on symmetric equilibria in which all farmers make the same decisions as each other.

(ii) Is it possible to normalise apples prices to 1?

(iii) Show that if the storage technology is perfect, then apples prices are equal in both seasons.

(iv) Show if the storage technology involves some spoilage, that apples are more expensive in Spring than Autumn.

(v) Suppose that the farmers' preferences have a discounted utility representation. (i.e. Time separable preferences that can be written in an additively separable fashion, with per-period utility functions being identical.) Moreover, assume that the farmers have decreasing marginal utility in apple and lettuce consumption. (a) Write the farmers' first-order conditions, (b) show that the farmers consume more apples in Spring than Autumn, and (c) write the farmer's problem using a Bellman equation.

(vi) Now suppose that one farmer is extra productive, and has double the endowments of all of the other farmers. The other farmers have a smaller endowment so that the aggregate endowments are identical. Think about the prices in the following scenarios:

   (a) The original symmetric equilibrium.
   (b) The new equilibrium (with the extra productive farmer).
   (c) A new equilibrium (with the extra productive farmer) in which the productive farmer is taxed so that the equilibrium allocation is the same as in (a).

   Do any of these scenarios share the same equilibrium prices?

(vii) Show that the farmers' second-period value function is concave and ** differentiable.

*Question* 6. (Micro 1 degree exam in May 2013) Suppose there are two countries of equal population. However, the big country has twice the amount of land, so that each household located there has twice the land endowment of households in the small country. Each country has an agricultural firm that transforms labour and land into food. Food can be traded on the international market. However, labour and land are more complicated. Each firm is owned equally by the citizens of its own country, and can only grow food on its own country's land. We say that workers migrate if they work for the other country's firm, although we assume that migration is costless.

(i) Write down a general equilibrium model of the labour, food and land markets. (Hint: treat labour and food as unified international markets, but land as national markets.)

(ii) Suppose that at some (out-of-equilibrium) prices, the food and labour markets clear, but there is excess demand of the small country's land. What does Walras' law say about the market for the large country's land?

(iii) Show that the small country's firm's profit function is convex in prices.

(iv) Show that if wages increase, the small country decreases its demand for labour.

(v) Show that if the production technology has constant returns to scale, and leisure is a normal good, then there is some migration from the small to the big country. (Hint: functions that are homogeneous of degree 1, i.e. satisfy the property that $f(tx, ty) = tf(x, y)$, also have the property that $f_x(2x, 2y) = f_x(x, y)$ for all $(x, y)$.)

(vi) Suppose the two countries plan to federalise into a free-trade zone (like the EU). They are worried about social tensions arising from the inequality of the people from the two countries. Devise a lump-sum tax scheme that creates perfect equality.

(vii) * Suppose that households are constrained to work in one country only (of their choice). Discuss how this possibility impedes application of the Brouwer's fixed point theorem to establish existence of equilibria.

*Question* 7. US comedian Lewis Black has the following to say about solar energy:

> If you ask your congressman why, he'll say "Because it's hard. It's really hard. Makes me want to go poopie." You know why we don't have solar energy? It's because the sun goes away each day, and it doesn't tell us where it's going!

Two countries are endowed with some electricity during the day time. However, they are located on opposite sides of the world, so when it is day time in one country, it is night time in the other. Electricity is non-storable, so the only way to consume electricity at night is to import electricity from the other country. A portion of the electricity is lost in transportation; the fraction lost increases as the amount of electricity transported increases.

Apart from this, the countries are identical: there is one household in each country, they share the same preferences and endowments, and the household in each country owns its own electricity exporter. You may assume preferences are additively separable across time, and they value electricity consumption equally during the day and night with decreasing marginal utility.

(i) Write down a general equilibrium model of this economy for one 24-hour period consisting of one night and day in each country. (Hint: treat electricity in different countries and different times as separate markets.)

(ii) It is possible to eliminate equilibrium variables and conditions using (i) price normalisation and (ii) Walras' law. Provide specific examples of how each of these may be done in the context of your model.

(iii) Suppose that both distributors discover a perfect transportation technology that prevents any electricity from being lost in transportation. In this case, show that both countries have the same sequence of electricity prices.

(iv) Show that if the distributors have a perfect transportation (as above), then the prices are the same. (Hint: look at the households' first-order conditions, and check the market clearing conditions.)

(v) Consider the proposal of taxing electricity consumption to subsidise electricity distributors to compensate them for the wasted energy lost. Would this proposal make everybody better off?

(vi) Again, suppose that there is a perfect transportation technology (see above). Consider the proposal of one country to invade the other, and to impose a new lump-sum tax on the victim country's household. The booty is distributed to the invading country's household. Does this make the invading household better off?

*Question* 8. (Micro 1 class exam in December 2013) Suppose there are two types of people: words people and numbers people. A medicine factory hires workers into two professions: marketing and engineering. Both types of people can do both types of jobs, but words people are better at marketing, and numbers people are better at engineering. Specifically, one hour of a words person's time spent on marketing is equivalent to two hours of a numbers person's time spent on marketing, and vice versa. Both types of people have the same preferences, and are indiffferent between both professions – they just take the best wage they can find. Everybody knows what type of person they are trading with.

(i) Define an equilibrium for this economy.

(ii) Suppose there is excess demand for both types of labour, i.e. at market prices, the firm demands more labour than the workers are willing to supply. Does this mean that there is also excess demand for medicine?

(iii) The factory has to make two types of choices: how many workers of each type to hire, and how to allocate them to professions.

  (a) Define the firm's output function as the maximum amount of medicine the firm can produce with given labour inputs.

  (b) Write down a Bellman equation for the factory relating the firm's cost function to the firm's output function.

  (c) Show that the firm's cost function is concave with respect to wages.

  (d) Show that if the market wage of numbers people increases, then the firm finds it optimal to meet its production target by hiring fewer numbers people and more words people.

(iv) Suppose the Words Union has an agreement which guarantees a maximum number of hours for words people only, and that this makes the words people better off. The Numbers Union proposes offering the Words Union a deal: it would tax numbers workers a little bit, and give those taxes to words workers. In return, the Words Union would abandon its maximum hours policy. Is it possible that both unions would agree to this deal?

(v) * Prove that the cost function is differentiable with respect to wages.

*Question* 9. (Micro 1 class exam in December 2013) A child care centre provides any number of hours of care to several households using two types of labour: babysitters and cleaners. Both types of labour are necessary for production – if either is zero, then no care can be provided. Households can simultaneously supply labour of both types. Households are also endowed with divisible houses, which they can exchange.

(i) Define the concept of a symmetric equilibrium for this economy, in which each household makes the same choice.

(ii) Suppose at all equilibrium allocations, the households have a higher marginal utility loss of cleaning than babysitting. Show that in every equilibrium, the cleaning wage is higher than the babysitting wage.

(iii) Suppose that the firm's production function is not concave. Does this imply that the profit function is not convex in prices?

(iv) Suppose that workers must specialise in at most one profession, babysitting or cleaning. (This isn't a government restriction, just a difficulty of working in these professions.) Are all equilibria efficient? Specifically, is it the case that every equilibrium in this environment is Pareto undominated by every feasible allocation in this environment?

(v) * As in the previous part, suppose that workers must specialise in at most one profession, babysitting or cleaning. Can every efficient allocation in this environment be implemented using lump-sum taxes?

*Question* 10. (Micro1 degree exam in May 2014) Suppose there are two rural districts that share an identical agricultural technology for transforming water into food. In the first year, households in both districts are endowed with the same amount of water, which they sell to farms. In the second year, one district suffers a perfectly predictable drought and has no water endowment. Households only directly consume food, and only hold shares in local farms. There are no import/export or migration costs, but food and water are non-storable.

(i) Write down a competitive general equilibrium model of the economy. You may assume households' preferences can be represented by an additively separable utility function.

(ii) Suppose that some protesters succeed in lowering the price of water in the second period, which leads to excess demand of water in the second period. According to Walras' law, what other consequences would this non-equilibrium behaviour have?

(iii) Show that each household has a decreasing marginal value of saving for the second year, provided that the household has a decreasing marginal utility of consumption. (Hint: this involves formulating the value of savings.)

(iv) Show that each household consumes less during the drought.

(v) The government would like to compensate the drought-striken district. Either devise a lump-sum tax policy that would implement smooth (constant) consumption over time for all households, or prove that this task is impossible.

(vi) * Write down a function that has the following property: a price vector is a fixed point of that function if and only if there exists an equilibrium with that price vector. Your function should never lead to negative prices. (You may make use of the excess demand function without defining it explicitly.)

*Question* 11. (Micro 1 degree exam in May 2014) Individuals are endowed with one unit of human capital and time. In the first year, individuals divide their time between accumulating human capital (through self-study), labour, and leisure. In the second year, the individuals divide their time between labour and leisure only. A firm produces a consumption good in each year using labour. The contribution of each hour of work to production is proportional to the worker's human capital.

(i) Write down a perfectly competetive model for this market. You may assume the households have additively separable utility, with stationary flow utility. (Hint: the human capital production function should have decreasing marginal product.)

(ii) Is it possible for the price of consumption in the first period to be 1?

(iii) Write down a value function for the start of the second year. (Hint: the state variable includes human capital, savings, and the prices in the second year.)

(iv) Derive the marginal value of (a) human capital and (b) savings.

(v) The government thinks that it's wasteful for everybody to become educated. It proposes a tax on labour earnings in the second year to encourage more labour to be supplied in the first year. Could such a policy be Pareto-improving?

(vi) * Informally discuss whether there are any asymmetric equilibria (e.g. in which some people choose to become well-educated, but others do not.)

*Question* 12. (Micro 1 class exam in December 2014) A factory produces appliances using labour and waste disposal services. Households supply labour and waste disposal. Households are endowed with small or large gardens, where they can dispose of waste. Assume that households do not suffer from storing waste in their gardens, and that gardens are not traded (or at least, not directly).

(i) Write down a competitive model of the labour, appliance, and waste disposal markets.

(ii) Show that in every equilibrium, all households' gardens are filled to capacity with waste.

(iii) Show that if leisure is a normal good, then households with bigger gardens work less.

(iv) Show that if the price of waste disposal increases, then firms will generate less waste.

(v) Suppose the government wants to decrease the amount of waste stored in gardens. Is there a lump-sum tax scheme that would work?

(vi) * Under what conditions would the households have a unique optimal labour, appliance and waste storage choice?

(vii) * Prove that if all prices are greater than zero, and that households can work at most 24 hours per day, then the budget set (i.e. the set of affordable feasible choices) is compact.

*Question* 13. (Micro 1 class exam in December 2014) As the earth's population grows, an important question is how future inhabitants will be able to feed themselves, and whether this will lead to inter-generational inequality. Suppose there are two generations (X and Y) of equal size. Generation X lives for two time periods, but Generation Y only lives in the second time period. This means that the population is higher in the second period.

Farms produce food using land and labour. Only Generation X is endowed with land, which it can supply to the market. Generation X households hold all of the shares in the farms. Both generations can supply labour and consume food. Households do not benefit from occupying land (but can gain wealth from renting out the land). Generation X has stationary time-separable preferences, and its per-period utility function is the same as Generation Y's.

(i) Write down a competitive general equilibrium model of this economy.

(ii) Suppose that if the prices in all markets (labour, land, and food) do not increase over time, that there is excess demand of labour, land, and food in the second period. Does this imply that there is excess supply in all markets in the first period?

(iii) For this part, focus attention on equilibria in which food output is higher in the second period. Show that in every such equilibrium, real wages (i.e. wages divided by food prices) are lower in the second period.

(iv) Write down Generation X's value of holding money in the second period. (Hint: this should be a function of money and second period food prices and wages.)

(v) Reformulate Generation X's problem by using the value function from (iv) twice, i.e. the household should choose how to allocate money between the two periods. How the money is spent in each period should be buried inside the value function.

(vi) Generation Y protestors would like to eat more and work less, so they propose confiscating land from Generation X at the start of period 2, and giving it to Generation Y. Can such a policy make Generation Y better off? Would the proposal lead Generation Y to eat more and work less?

(vii) * The proof of existence of equilibrium relies on applying Brouwer's fixed point theorem, which requires a set to be convex (among other things). Economically speaking, which set is convex? Is this assumption usually met?

(viii) * Holding prices fixed, consider a sequence of optimal labour supply and consumption choices, where the expenditure decreases to 1. Does this sequence have a convergent subsequence (using the Euclidean metric)?

*Question* 14. (Micro 1 degree exam in May 2015) Suppose there are two occupations, nursing and cleaning, and that individuals must select only one occupation to work in each year. Cleaning is easy to learn, but nurses with one year of experience become more productive. There are two years in the economy. Hospitals hire nurses and cleaners to provide medical services, and share their profits equally among the population. Individuals consume medical services.

(i) Write down a competitive model of the nursing and cleaning markets across the two years. (Hint: there are no symmetric equilibria, so you will need to accommodate identical households taking different decisions.)

(ii) Write down a formula for the value of savings and nursing experience in the second year.

(iii) Reformulate the year-one households' problem using the value function from the previous part.

(iv) What is the marginal value of nursing experience if the individual finds it optimal to do cleaning in the second year?

(v) Argue informally that nurses have lower wages than cleaners in the first year.

(vi) Are competitive equilibria Pareto efficient in this economy? (Hint: list all the differences from pure-exchange economies where we proved the first-welfare theorem, and informally discuss whether these are important.)

(vii) * Is the excess demand function continuous?

(viii) ** Is the household's feasiable choice set compact, assuming all prices are strictly greater than zero?

*Question* 15. (Micro 1 degree exam in May 2015) Suppose there are two schools that hire workers to teach. One school is twice as productive as the other – i.e. for the same amount of input, it produces double the output. Households supply labour and consume education.

(i) Write down a competitive model of this economy.

(ii) Suppose at prevailing prices, there is excess supply of teachers. What does this imply about the supply of education?

(iii) Prove that the "good" (more productive) school hires more teachers than the "bad" school.

(iv) Prove that if wages increase, then schools provide less education.

(v) Suppose that the government imposes lump-sum taxes on half of the population, and transfers these to the other half equally. Moreover suppose that education and leisure are normal goods, and that this policy causes real wages to increase. What happens to each household's education choices? Hint: the Slustky equation is:

$$\underbrace{\frac{\partial x_i(p,m)}{\partial p_j}}_{\text{net effect}} = \underbrace{\left[\frac{\partial h_i(p,u)}{\partial p_j}\right]_{u=v(p,m)}}_{\text{substitution effect}} + \underbrace{\underbrace{-x_j(p,m)}_{\text{wealth lost}} \frac{\partial x_i(p,m)}{\partial m}}_{\text{income effect}}. \tag{10}$$

(vi) * In class, to prove the existence of an equilibrium, we constructed a continuous function and proved that it has a fixed point. Since we only need to consider one price in this economy (why?), this function effectively maps from $\mathbb{R}$ to $\mathbb{R}$. Describe mathematically, and sketch (i.e. draw) this function.

(vii) ** Let $(X, d)$ be any metric space. Prove that if $f, g : X \to \mathbb{R}$ are continuous, then $h(x) = \max\{f(x), g(x)\}$ is also continuous. Hint: you may assume a similar result holds for addition and subtraction.

*Question* 16. (Micro 1 class exam in December 2015) Consider a two-generation economy in which both generations consume fish in both time periods. However, the old generation can only work in the first period and the young can only work in the second period. A fishing firm hires workers in each period to catch fish, and a storage firm hires workers to freeze fish in the first time period, and to defrost fish in the second period. Defrosted and fresh fish are perfect substitutes.

(i) Write down a competitive model of the intergenerational fishing economy.

(ii) Is it possible to normalise real wages in the first period to 1?

(iii) Show that if the price of fish in the second period increases, the storage firm sells more fish.

(iv) The government is worried about intergenerational inequality, i.e. that the young will receive lower real wages than the old. It proposes a lump-sum tax on the old and transfer to the young. Show if leisure is a normal good, then this causes at least some prices to change in the new equilibrium.

(v) Suppose it is only possible to store whole fish. Are all equilibria Pareto efficient?

(vi) * Suppose households can home-produce fish storage. Give an example of how this might lead household preferences to be time-inseparable.

(vii) ** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Prove that if $f : X \to Y$ is continuous and $X$ is compact in $(X, d_X)$, then $f(X)$ is compact in $(Y, d_Y)$.

*Question* 17. (Micro 1 class exam in December 2015) Suppose that there are two time periods, and two seasons – summer and winter. There are about ten times as many people in the northern hemisphere than the southern hemisphere. This means that in both periods, an unequal fraction of people experience summer and winter. People prefer to work less and consume more in summer. A firm hires workers to produce a consumption good. It operates in both periods.

(i) Write down a competitive equilibrium model of seasons and hemispheres.

(ii) Suppose the market value of excess demand in all markets in the first time period is positive. Does this mean that there must be excess supply in a market in another time period?

(iii) Using dynamic programming, reformulate the households' problems using net borrowing/lending as a state variable. That is, if this state variable is a positive number for period 1, then the household consumes more than its wages in period 1. The Bellman equation should bury the specifics about consumption or labour decisions in *both* periods.

(iv) Show that households have a decreasing marginal value of net borrowing.

(v) Show that households do more net borrowing (or less net lending) in summer than winter. *Hint: treat "how 'northern' a household is" as a state variable.*

(vi) The United Nations is worried that because of the population imbalance, the seasons create global inequality. They propose achieving equality by requiring everyone to work the same hours during summer and winter. Is it possible to design a lump-sum tax scheme that implements such an allocation? *Hint: assume that leisure is a normal good.*

(vii) ** Prove that the boundary $\partial A$ of any set $A$ is closed.

*Question* 18. (Micro 1 degree exam in May 2016) Scotland has two major cities, Glasgow and Edinburgh. Suppose that each city has an identical stock of buildings. Workers prefer to consume more buildings, and only benefit from housing located in the city that they choose to work in. There is an electronics factory in each city, that uses labour and buildings to produce electronics. The Glasgow factory is $z > 1$ times as productive as the Edinburgh factory (given the same inputs). To summarise, workers supply labour to factories, consume housing services in their own city, and consume electronics.

(i) Write down a competitive model of the Scottish housing and electronics economy.

(ii) Suppose that there were excess demand for workers and housing in Glasgow, and that the electronics market cleared. Does this imply that there would be excess supply of workers and/or housing in Edinburgh?

(iii) Prove that the Glasgow manufacturer's profit is increasing and convex in its productivity $z$.

(iv) Prove that if wages in Glasgow increase, then the Glasgow manufacturer demands fewer workers.

(v) Prove that if wages are higher in Glasgow, then rent is also higher in Glasgow.

(vi) Suppose there are several equilibria. Prove that every worker is indifferent between all equilibria.

(vii) * Prove that there is only one equilibrium allocation of resources.

(viii) ** Prove that if $f$ and $g$ are continuous, then $h(x) = f(g(x))$ is continuous.

*Question* 19. (Micro 1 degree exam in May 2016) According to Seixas, Robins, Attfield and Moulton (1992), coal miners have a 16% risk of developing the disease *black lung*. To keep things simple, suppose that all coal workers must retire early because of their health. Specifically suppose there are two time periods, and workers can choose to work in call centres or coal mines each period. After working in a coal mine, the worker is unable to work thereafter (in any job). However, sick retirees can still enjoy leisure as normal. A firm sells electricity, which it produces with coal miners and call centre workers. Workers supply either kind of labour and consume electricity and leisure.

(i) Write down a competitive model of the electricity markets and the two types of labour markets.

(ii) Reformulate the worker's problem with a Bellman equation, using wealth and health as state variables.

(iii) Prove that in the last period, both professions receive the same wage.

(iv) Prove that the worker has diminishing marginal value of wealth in the last period.

(v) Prove that in the first period, coal miners receive higher wages than call centre workers.

(vi) Suppose the government selects half of the population (e.g. those born in the first half of the year) for a reward, to be funded by lump-sum taxes on the other half of the population. Is this policy Pareto efficient?

(vii) ** Consider the metric space $(X, d)$ where $X = [0, 2]$ and $d(x, y) = |x - y|$. Prove or disprove that $A = [0, 1)$ is an open set.

*Question* 20. (Advanced Mathematical Economics mock exam)

**Part A.** Parts (i), (ii), (iii), and (iv) of Question 19.

**Part B.**

(i) Let $X = \{(x, y) \in \mathbb{R}^2 : x + y \leq 1\}$. What is the boundary of the set

$$A = \left\{(x, y) \in \mathbb{R}^2_+ : x + y \leq 1\right\}$$

inside the metric space $(X, d_2)$?

(ii) Consider the sequence of functions $f_n \in CB([0, 1])$ defined by $f_n(x) = x + x/n$. Is $f_n$ a convergent sequence in $(CB[0, 1], d_\infty)$?

(iii) Prove that $(l_\infty([0, 1]), d_\infty)$ is not a compact metric space. (Recall that $l_\infty([0, 1])$ is the set of bounded sequences $x_n \in [0, 1]$.) *Hint: you only need to find one counterexample.*

(iv) Consider any metric space $(X, d)$. Let $x_n, y_n, z_n \in X$ be sequences. Suppose $x_n \to x^*$ and $z_n \to x^*$. Prove that if $d(x_n, y_n) \leq d(x_n, z_n)$ for all $n$ then $y_n \to x^*$.

(v) Write down a recursive Bellman equation for an infinite horizon cake-eating problem in which the size of the cake grows by $r = 0.01 \times 100\%$ every day. Prove that the Bellman operator a contraction on $(CB(\mathbb{R}), d_\infty)$. What is the degree of the contraction? (You do not need to prove that the Bellman operator is a self-map.)

(vi) Let $(X, d)$ be a complete metric space, and $f : X \to X$ be a continuous function. Fix any $x_0 \in X$, and consider the sequence $x_{n+1} = f(x_n)$. Prove that if $x_n$ is a Cauchy sequence then $x_n$ converges to a fixed-point of $f$.

(vii) Let $X = \{f \in CB([0, 1]) : f(x) = ax \text{ for some } a \in [0, 1]\}$. Is $(X, d_\infty)$ a compact metric space?

(viii) Prove that the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at $x^* = 0$.

*Question* 21. (Advanced Mathematical Economics December 2016)
**Part A**

CAF (Construcciones y Auxiliar de Ferrocarriles) produces trams and replacement parts for Edinburgh Trams using labour. Suppose that for each tram used in the first year of operation, 0.2 trams worth of parts must be bought for maintenance before the tram can be used in the second year. Edinburgh Trams produces public transport services from trams and labour to households. Households supply labour to the two companies and consume transport. All households have the same preferences, and shares in all firms are shared equally among all households.

(i) Write down a general equilibrium model of the labour, tram and transportation markets involving households, the factory, and the tram operator over a two-year period. (Hint: Pay careful attention to the depreciation of trams.)

(ii) Write down a Bellman equation for Edinburgh Trams' decision in the first year that buries the second year choices in a value function.

(iii) Show that Edinburgh trams' second year value function is convex in prices.

(iv) Show that if the price of trams increases in the second year, then Edinburgh Trams buys fewer trams in the second year.

**Part B**

(i) Let $(X, d)$ be a metric space and let $A \subseteq X$. Prove that the boundary of $A$ is a closed set.

(ii) Suppose $(X, d)$ is a compact metric space. Prove that if $A \subseteq X$ is a closed set, then $A$ is a compact set.

(iii) Let $(A, d)$ be a compact metric space. Consider an optimisation problem:

$$\max_{a \in A} u(a),$$

where $u : A \to \mathbb{R}$ is continuous. Prove that the set of optimal choices,

$$A^* = \{a \in A : u(a) \geq u(a') \text{ for all } a' \in A\}$$

is compact. *Hint: use the previous question.*

(iv) Prove that $(CB(\mathbb{R}), d_\infty)$ is *not* a compact metric space. *Hint: you only need one counterexample.*

(v) Suppose that the stock of salmon in the North Sea naturally doubles every five years. Individuals enjoy eating salmon according to a discounted utility function. (a) Write down a recursive Bellman equation to represent the social planner's problem over an infinite time horizon. (b) Sketch a proof that the social value of the stock of salmon is a continuous function. (You do not need to prove that the Bellman operator is a contraction, or prove the principle of optimality.)

(vi) Let $f : X \to X$ be a function on the metric space $(X, d)$. Prove that if $f$ has two fixed points, $x^* \neq x^{**}$, then $f$ is not a contraction.

(vii) Let

$$u(x, y) = \frac{x + y}{1 + y^2 - \sqrt{y}},$$

where $(x, y) \in \mathbb{R}_+ \times [0, 1]$. Find a differentiable lower support function at $x = 2$ for

$$f(x) = \max_{y \in [0,1]} u(x, y).$$

(viii) Suppose that $f : \mathbb{R}_+^{N-1} \to \mathbb{R}$ is strictly concave. Prove that there is at most one solution to the profit maximisation problem,

$$\max_{x \in \mathbb{R}_+^{N-1}} pf(x) - w \cdot x$$

where $(p, w) \in \mathbb{R}_{++}^N$.

*Question* 22. (Microeconomics 1, December 2016) According to the Lincoln Longwool Sheep Breeders Association, the Lincoln Longwool sheep is "one of the most important breeds ever seen in our green and pleasant land." It is a "dual-purpose" breed, meaning it yields high quality wool and meat. Suppose that sheep live for up to two years. If a sheep is killed at the end of the first year, it yields both wool and meat. If a sheep is killed at the end of the second year, it yields wool in both years and the same amount of meat. Households are endowed with sheep, and consume meat and wool each year. Households' preferences can be represented with a discounted utility function. Farms buy sheep to produce wool and meat.

(i) Write down a competitive model of the sheep, wool and meat markets across the two years.

(ii) Prove that farms demand more sheep in the first year if the price of sheep decreases (but no other prices change).

(iii) Write down the firm's value of owning live sheep in the first and second years, making use of a Bellman equation. Prove that these are concave functions of the number of sheep.

(iv) Find an assumption on the model parameters such that the price of sheep decreases over time.

(v) Suppose that half of the population is poor, and only has only half of the sheep endowment. Is it possible to devise a lump-sum transfer scheme that institutes equal welfare for each household?

(vi) * Let $X = \mathbb{R}^6_+$. Suppose there is a continuous function $f : X \to X$ with the properties that (1) $p \in X$ is an equilibrium price vector if and only if $f(p) = p$ and (2) $f(tx) = f(x)$ for all $t > 0$. (a) Apply Brouwer's fixed point theorem to prove that an equilibrium exists. *Hint: you will need to reformulate $f$.* (b) Fix any $p_0 \in X$. Without using Brouwer's point theorem, prove that if the sequence $p_{n+1} = f(p_n)$ is a Cauchy sequence, then $f$ has a fixed point.

*Question* 23. (Microeconomics 1, December 2016) Suppose a country consists of workers with and without university degrees. Only university graduates can design machines, but both types of worker are equally competent at operating machines. There are two firms: a machine manufacturer that hires university graduates and a clothing manufacturer that buys machines and can hire either type of worker. Workers sell labour and consume clothing and machines (for washing their clothes).

(i) Formulate a competitive equilibrium model of the markets for both types of labour, machines and clothing. *Hint: do not assume that equilibria are symmetric.*

(ii) Suppose at some market prices, the supply of university-educated labour exceeds demand. Does this imply that the demand for uneducated labour exceeds supply?

(iii) Suppose the two firms decide to merge into single firm. (a) Write the combined-firm's profit-maximisation problem using a Bellman equation to separate the output and input choices. (b) Does the equilibrium (or equilibria) change after the merger?

(iv) Prove that if the wages of uneducated workers increases, the clothing manufacturer hires fewer uneducated workers.

(v) Prove that if the clothing manufacturer hires educated workers, then the wages paid to all workers by both firms are equal.

(vi) Suppose every Pareto efficient allocation involves university graduates working for the machine manufacturer only. Is it possible to find lump-sum transfers to implement an allocation in which some university graduates work for the clothing manufacturer?

(vii) * Let $X = \mathbb{R}^4_+$. Suppose there is a continuous function $f : X \to X$ with the properties that (1) $p \in X$ is an equilibrium price vector if and only if $f(p) = p$ and (2) $f(tx) = f(x)$ for all $t > 0$. (a) Apply Brouwer's fixed point theorem to prove that an equilibrium exists. *Hint: you will need to reformulate $f$.* (b) Fix any $p_0 \in X$. Without using Brouwer's point theorem, prove that if the sequence $p_{n+1} = f(p_n)$ is a Cauchy sequence, then $f$ has a fixed point.

*Question* 24. (Advanced Mathematical Economics, May 2017 exam)

**Part A**

Until plastic bottles became popular in the 1960s, milk was sold in glass bottles that could be reused. For simplicity, assume there are two time periods. Suppose households supply labour, and buy bottled milk and empty bottles in both periods. Milk bottles from the first period become empty in the second period, and households can sell these (or buy even more). A firm uses labour to make bottles and bottled milk in both periods.

(i) Write down a competitive model of the bottled milk industry.

(ii) Reformulate the firm's problem by separating the firm's milk and bottle production decisions. *Hint: this is a bit like dynamic programming, but the "Bellman equation" has no choice variables.*

(iii) Prove that the firm has an increasing marginal profit (i.e. a decreasing marginal loss) of a second-period wage increase.

(iv) Prove that the firm reacts to a second-period bottle price increase by increasing its net supply of (empty and filled) bottles.

**Part B**

(i) Consider the metric space $(X, d_2)$ where $X = [0,1] \times \mathbb{R}$ and $d_2(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. What is the boundary of the set $A = [0,1] \times \{0\}$ in this space?

(ii) Let $X = \{f : [0,1] \to \mathbb{R}$ s.t. $f$ is continuously differentiable$\}$ and

$$d(f,g) = d_\infty(f,g) + d_\infty(f', g'),$$

where $f'$ and $g'$ are the derivatives of $f$ and $g$ respectively, and $d_\infty(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$. Prove (a) $d$ is well-defined and (b) $(X, d)$ is a metric space.

(iii) Consider the metric space $(X, d_1)$ where $X = (0,1)$ and $d_1(x,y) = |x - y|$. Supply a counter-example to prove that $(X, d_1)$ is *not* complete.

(iv) Consider the metric space $(X, d_1)$, where $X \subseteq [0,1]$ and $d_1(x,y) = |x - y|$. Suppose that $x_n \in X$ has no convergent subsequence. Prove that $X$ is not a closed set in $(\mathbb{R}, d_1)$.

(v) Let $x_t \in [0,1]$ be the fraction of the population of generation $t$ that is religious. Suppose that each subsequent generation's demographics are deterministic with $x_{t+1} = f(x_t)$, and that $x_t \to x^*$. Prove that if $f$ is a continuous function, then $x^*$ is a fixed point of $f$, i.e. $x^*$ is a steady state.

(vi) Prove that $f(x) = \frac{1}{3}x^2$ is a contraction on the metric space $(X, d) = ([0, 1], d_1)$ where $d_1(x, y) = |x - y|$.

(vii) Consider a two player-game where player one and two choose $a \in [0, 1]$ and $b \in [0, 1]$ respectively. Suppose that player one and two have best response functions $f(b)$ and $g(a)$ respectively. Let $X = A \times B$ and $h : X \to X$ be defined by $h(a, b) = (f(b), g(a))$. Consider the following procedure (called iterated deletion of dominated strategies) for calculating Nash equilibria:

(a) Set $Y_1 = X$.

(b) Let $Y_{n+1} = h(Y_n)$, that is $Y_{n+1} = \{h(a, b) : (a, b) \in Y_n\}$.

(c) Report $Y_\infty = \cap_{n=1}^{\infty} Y_n$.

Prove that if $h$ is continuous, then $Y_\infty \neq \emptyset$, i.e. that this procedure does not delete all strategies. *Hint: Apply the Cantor intersection theorem.*

(viii) Recall that $CB(\mathbb{R}_+)$ is the set of continuous and bounded functions with domain $\mathbb{R}_+$ and co-domain $\mathbb{R}$, whose distances can be measured with the metric

$$d_\infty(f, g) = \sup_{x \in \mathbb{R}_+} |f(x) - f(y)|.$$

Consider the following Bellman operator $\Phi : CB(\mathbb{R}_+) \to CB(\mathbb{R}_+)$, which is a contraction of degree $\beta$ on $(CB(\mathbb{R}_+), d_\infty)$:

$$\begin{aligned} \Phi(V)(k) = &\max_{c,k'} u(c) + \beta V(k') \\ &\text{s.t. } c + k' = g(k). \end{aligned}$$

(You may interpret $c$ as consumption, $k$ as capital, $g(k)$ as output $u(c)$ as utility, and $\beta$ as the rate of time preference.) Use Banach's fixed point theorem to prove that if $u$ and $g$ are concave, then the fixed point of $\Phi$ is concave.

*Question* 25. (Microeconomics 1, May 2017) Consider an economy with two time-periods, in which the entire population lives for both periods. The young and old are identical, except the young have no labour endowment in the first period. They can supply up to their labour endowment and consume food in each period, and have time-separable preferences. A farm produces food from labour.

(i) Devise a competitive model of the food and labour markets.

(ii) Suppose that at the (non-equilibrium) market prices, the market values of the excess demands for food sum to a positive number. Prove that there is excess supply in at least one of the labour markets. *Note: do not assume that there is excess demand in both food markets.*

(iii) Prove that the farm reacts to second-period food-price increases by increasing supply.

(iv) Write down the utility maximisation problem of a "big family" household that makes all market transactions on behalf of the households and the farm. Assume that the big-family household puts equal utility weight on all actual households. *Hints. Recall the home-production example from class. Put the market transactions in one Bellman equation, put the farm choices inside another Bellman equation, and bury the allocation of resources to households inside a value function.*

(v) Suppose the government forcibly reallocated all resources to an efficient egalitarian allocation. If the population were allowed to trade based on their new endowments, what competitive allocation would arise?

(vi) * Give an example of a metric space with the property that every closed subset is compact.

(vii) * Prove the Cantor intersection theorem:

Let $(X, d)$ be a metric space. Suppose $A_n \subseteq X$ is a sequence of sets such that (a) $A_{n+1} \subseteq A_n$, (b) $A_n \neq \emptyset$ and (c) $A_n$ is compact for all $n$. Let $A = \cap A_n$. Then $A \neq \emptyset$.

*Question* 26. (Microeconomics 1, May 2017) A café and a restaurant both serve meals to customers, using labour and food. The restaurant requires double the labour and food inputs to produce the same number of meals as the café. Households supply labour and only eat at restaurantes and/or cafes. At every level of consumption and supply, households prefer an extra restaurant meal to an extra café meal. A farm produces food from labour only.

(i) Write down a competitive equilibrium model of the labour, food, and meals (restaurants and cafes) markets.

(ii) Suppose there is an equilibrium in which restaurant meals cost £1. Does this mean that there is an equilibrium in which café meals cost £1?

(iii) Prove that in every equilibrium in which café meals are sold, restaurant meals trade at a higher price than café meals.

(iv) Prove that the marginal cost of restaurant meals equals the wage divided by the marginal productivity of labour.

(v) Prove that the restaurant's marginal cost curve is increasing.

(vi) The goverment would like to increase restaurant meal consumption. It proposes (symmetric) lump-sum transfer scheme from households to the restaurant. Would this policy have the desired effect?

(vii) * Give an example of a metric space with the property that every closed subset is compact.

(viii) * Prove the Cantor intersection theorem:

Let $(X, d)$ be a metric space. Suppose $A_n \subseteq X$ is a sequence of sets such that (a) $A_{n+1} \subseteq A_n$, (b) $A_n \neq \emptyset$ and (c) $A_n$ is compact for all $n$. Let $A = \cap A_n$. Then $A \neq \emptyset$.

*Question* 27. (Advanced Mathematical Economics, December 2017) **Part A**

Internet data centres generate waste energy that can be used to heat homes. Suppose that this waste energy can be transported but not stored. Households benefit more from heat during the evening, and benefit more from the Internet during the day. Households own the data centres, which they rent out to the internet company.

(i) Write down a competitive equilibrium model of the data centre, computing and heat markets during the day and evening.

(ii) Prove that if the daytime heating price increases, then the firm sells more daytime internet services.

(iii) Write down a Bellman equation that separates the household's problem into day and evening choices.

**Part B**

(i) Let $(X, d_X)$ and $(Y, d_Y)$ be complete metric spaces. Let $Z = X \times Y$ and $d_Z((x, y), (x', y')) = \max \{d_X(x, x'), d_Y(y, y')\}$. Prove that $(Z, d_z)$ is a complete metric space.

(ii) Let $(X, d)$ be any metric space, and $A \subseteq X$ any subset. Provide a counter-example to the following false statement: the interior of the boundary of $A$ is empty, i.e. $\text{int}(\partial A) = \emptyset$.

(iii) Prove that if $x$ is a boundary point of $A$ in $(X, d)$ (defined in terms of sequences), then every open neighbourhood $U$ of $x$ has $U \cap A \neq \emptyset$ and $U \cap (X \backslash A) \neq \emptyset$.

(iv) Prove that $f : X \to Y$ is continuous if and only if for every open ball $U = N_r(y)$, the inverse image $f^{-1}(U)$ is an open set.

(v) Suppose $f : \mathbb{R} \to \mathbb{R}$, where distances in both the domain and co-domain are measured with the Euclidean metric. Suppose that $\lim_{n\to\infty} f(1/n) = 1$ and $f(0) = 0$. Provide an example of an open set $U$ such that $f^{-1}(U)$ is not open.

(vi) Let $x_t$ be the fraction of women that work in professional occupations. Assume that this changes over time according to $x_{t+1} = f(x_t)$ where $f : [0, 1] \to [0, 1]$, and distances are measured by the Euclidean metric. Now, suppose that (i) $f$ is continuous, and that (ii) $f$ is a contraction on $[0, \frac{1}{3})$, and also on $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, 1]$. Prove that there are either two or three steady-states (i.e. fixed points of $f$).

(vii) Investors with £200000 of assets are able to acquire visas (under some other conditions) to migrate to the United Kingdom. People residing outside the UK receive labour income $w$ each period, and choose how much to consume $c$ and save $a'$, and whether to migrate to the UK. Their utility each period is $u(c)$, which is discounted at rate $\beta$. Let $M(a)$ be the value of living in the UK as a migrant with assets $a$. Both $u$ and $M$ are bounded and concave. The value of assets to a foreigner $V(a)$ is characterised by the Bellman equation

$$V(a) = \begin{cases} M(a) & \text{if } a \geq 200000 \\ \max_{a'} u(a + w - a') + \beta V(a') & \text{if } a \in [0, 200000). \end{cases}$$

You hope to prove that $V$ is concave with the following strategy – which turns out not to work:

(a) Prove that the Bellman operator is a contraction in $(B(\mathbb{R}_+), d_\infty)$.

(b) Prove that $(X, d_\infty)$ is a complete metric space, where

$$X = \{V \in B(\mathbb{R}_+) : V \text{ is concave and } V(a) = M(a) \text{ for } a \geq 200000\}.$$

(c) Prove that the Bellman operator is a self-map on $X$.

(d) Apply Banach's fixed point theorem.

Which step(s) succeed and which step(s) fail? *You will get credit for checking as many steps as you can.*

(viii) Prove that the following optimization problem (relating to moral hazard) has an optimal solution:

$$\begin{aligned} &\min_{a,b \in \mathbb{R}_+} p \exp(a) + (1-p) \exp(b) \\ &\text{s.t. } pa + (1-p)b \geq 1, \text{ and} \\ &a - b \geq q, \end{aligned}$$

where $p \in (0, 1)$ and $q > 0$.

*Question* 28. (Microeconomics 1, December 2017)

Several identical households enjoy ice cream more in summer than winter, and enjoy soup in winter more than summer. Households are endowed with cows and fishing boats. Ice cream is made from cows. Soup is made from fishing boats. There are only two time periods (winter and summer).

(i) Formulate a competitive equilibrium model of cows, boats, ice cream and soup during summer and winter.

(ii) Suppose that the boat, cow, and icecream markets clear. Does this imply that the soup markets clear?

(iii) Reformulate the households' problem by constructing a value function for both time periods, which are connected via a Bellman equation.

(iv) How does the winter supply of icecream change when the winter price of soup increases?

(v) Is the Pareto frontier of this economy a convex set (under appropriate convexity assumptions about preferences and production)?

(vi) Prove that there is only one competitive equilibrium (under appropriate convexity assumptions about preferences and production).

(vii) * Consider any metric space $(X, d)$, and any two sets $A$ and $B$ with $A \subseteq B \subseteq X$. Prove that if $A$ is open in $(X, d)$, then $A$ is open in $(B, d)$.

(viii) * Let $f : X \to X$ be a function on a complete metric space $(X, d)$. Suppose that $g(x) = f(f(x))$ is a contraction. Prove that $f$ has a unique fixed point.

*Question* 29. (Advanced Mathematical Economics, May 2018)
**Part A**
Are improvements in renewable energy technology good news for climate change?

Suppose households own oil deposits, which they can extract and sell at any time. In the first year, only the oil-based power firm operates; it buys oil from households. In the second year, the solar-based power firm is able to hire workers to run solar plants. A pharmaceutical firm hires workers and uses electricity to make medicine. Households only consume electricity and medicine.

(i) Write down a competitive equilibrium model of the power and pharmaceutical industries.

(ii) Prove that the oil-based firm reacts to an energy price decrease in the second year (keeping all other prices fixed) by buying less oil in the second year.

(iii) How does the nature of the solar-based power firm's production function affect the equilibrium amount of oil extracted?

(iv) Decompose the pharmaceutical firm's choices into input and output choices using a Bellman equation.

**Part B**

(i) What is the interior of $A = \mathbb{Q}^2$ inside the metric space $(\mathbb{R} \times \mathbb{Q}, d_2)$? Recall that $\mathbb{Q}$ is the set of rational numbers, and the Euclidean metric on this space is $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

(ii) Consider the metric space $(\ell_\infty(\mathbb{R}), d_\infty)$, i.e. the set of sequences whose absolute values sum to a finite number. Provide an example of a contraction $f : \ell_\infty(\mathbb{R}) \to \ell_\infty(\mathbb{R})$. Recall that $d_\infty(\{x_n\}, \{y_n\}) = \sup_{n\in\mathbb{N}} |x_n - y_n|$.

(iii) Consider any metric space $(X, d)$ and any function $a : X \to \mathbb{R}_{++}$. Let $F_a(X) = \{f : X \to \mathbb{R}, \sup_{x\in X} a(x)f(x) < \infty\}$ and

$$d_a(f, g) = \sup_{x\in X} a(x) d_2(f(x), g(x)).$$

Prove that $(F_a(X), d_a)$ is a metric space. (Boyd, 1990, Journal of Economic Theory used this space to study unbounded value functions.)

(iv) Prove that the metric space $(F_a(X), d_a)$ as defined in the previous question is complete.

(v) Suppose that a person of height $h \in [0,1]$ has a utility function for food consumption $c \in [0,2]$ of $u_h(c) = c^h$. Prove that the set of these utility functions, $U = \{u_h : h \in [0,1]\}$ is a compact subset of the metric space $(CB([0,2]), d_\infty)$.

Note: you can assume that $f(x,y) = x^y$ and similar functions are continuous. Recall that $CB([0,2]) = \{f : [0,2] \to \mathbb{R}, f \text{ is continuous and bounded}\}$, and $d_\infty(f,g) = \sup_{x \in \mathbb{R}_+} |f(x) - g(x)|$.

(vi) Let $(X,d)$ be a non-empty compact metric space, and consider any continuous utility function $u : X \to \mathbb{R}$. Let $X^*$ be the set of optimal choices, i.e. $X^* = \operatorname{argmax}_{x \in X} u(x)$. Prove that $X^*$ is non-empty and compact.

(vii) Consider the optimisation problem:

$$\max_{a,b \in \mathbb{R}_+} p(x-a) + (1-p)(y-b)$$
$$\text{s.t. } u(a,e) \geq u(b,0) \text{ and } v(b,0) \geq v(a,e),$$

where $u$ and $v$ are continuous functions, $p \in [0,1]$ and $x, y \in \mathbb{R}_+$. Assume that there is a feasible choice $(\bar{a}, \bar{b})$ that satisfies both constraints. Prove that there exists an optimal choice $(a^*, b^*)$.

The economic content of this model – which is not necessary for solving the problem – is as follows. A proportion $p$ of the workers are "good", i.e. they have utility function $u$, not $v$, and they produce output $x$, not $y$. A recruiter wants to hire an optimal mix of good and bad workers. But he can't tell them apart. Instead, students can put effort $e$ into their education, which the recruiter can observe. So, the recruiter selects wages $a$ for the highly educated and $b$ for lowly educated students.

(viii) Consider the Bellman equation of a firm that makes a profit of $\pi(n)$ when it has $n \in \mathbb{N}$ workers, but it costs $\Delta(n,n')$ to hire (or fire) $n' - n$ workers tomorrow when it has $n$ workers today:

$$V(n) = \max_{n' \in \mathbb{N}} \pi(n) - \Delta(n,n') + \beta V(n').$$

Assume that $\pi$ is bounded, $\Delta(n,n) = 0$, $\Delta(n,n') \geq 0$ and $\Delta(n,\cdot)$ is unbounded. Prove that the Bellman equation has exactly one solution.

*Question* 30. (Microeconomics 1, May 2018) Suppose that Idaho farmers each own a field of Russet potatoes, and North Carolina farmers each own a field of sweet potatoes. Assume there are an equal number of farmers in Idaho and North Carolina. All farmers have the same preferences. In this question, do *not* assume that potatoes are (for all prices) normal goods.

(i) Formulate a pure-exchange competitive model of the sweet potato and Russet potato markets.

(ii) Use dynamic programing to reformulate the farmers' utility maximization problem. Specifically, write a Bellman equation that connects the indirect utility function (that gives each farmer's value as a function of prices and endowments) to the expenditure function (that gives each farmer's net expenditure as a function of prices, endowments, and a utility target).

(iii) Suppose that a particular price level, Idaho farmers respond to a price increase in Russet potatoes by consuming less Russet potatoes. Prove that this implies that Russet potatoes are normal goods for Idaho farmers. *Recall that a good $X$ is an inferior good if the demand for $X$ decreases when the wealth of the consumer increases. Hint 1: apply the envelope theorem to the expenditure function. Hint 2: you may find the Slutsky equation from the lecture notes helpful:*

$$\underbrace{\frac{\partial x_i(p,m)}{\partial p_j}}_{\text{net effect}} = \underbrace{\left[\frac{\partial h_i(p,u)}{\partial p_j}\right]_{u=v(p,m)}}_{\text{substitution effect}} + \underbrace{\underbrace{-x_j(p,m)}_{\text{wealth lost}}\frac{\partial x_i(p,m)}{\partial m}}_{\text{income effect}}.$$

(iv) *For the rest of this question, suppose that your model has two equilibria (while holding all model parameters fixed, including endowments): one in which both types of farmer consume the same things, and one in which North Carolina farmers consume more of both types of potato than Idaho farmers.*

Which of these two equilibria do North Carolina farmers prefer?

(v) Sketch and explain a graph showing a possible shape of the excess demand function of sweet potatoes as a function of the price of sweet potatoes.

(vi) For each of the two equilibria, devise a lump-sum tax policy that implements that equilibrium.

(vii) * Suppose $(X, d_X)$ and $(Y, d_Y)$ are non-empty metric spaces, and consider the metric space $(Z, d_Z) = (X \times Y, d_Z)$ where

$$d_Z(x, y; x', y') = \max \{d_X(x, x'), d_Y(y, y')\}.$$

Prove that if $(Z, d_Z)$ is a compact metric space, then $(X, d_X)$ is a compact metric space.

(viii) * Let $(X, d)$ be a complete metric space, and let $A \subseteq X$. Suppose that $f : X \to X$ is a contraction, and that $f(A) \subseteq A$. Prove that $f$ has a fixed point $x^*$ that lies in the closure of $A$.