Practice Questions

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Notes for Microeconomics 1 students. My part of the class and degree exams have an identical format and marking scheme, approximately as follows (not counting the bonus questions). You will be rated as no/almost/ok/good/excellent (i.e. 0 to 4) on the following learning outcomes:

- Formulating a model. Excellence here means the absence of important mistakes.
- Breadth of technique: Walras law, dynamic programming, the envelope theorem, convex analysis, the first welfare theorem, etc. Excellence here usually means applying four techniques.
- Depth of technique: using a technique in an unusual way, combining several techniques to deduce something, or a clever piece of logic. For example, proving that there is at most one equilibrium in a particular model by combining the first welfare theorem with symmetry of all households. Obviously, depth requires at least some breadth, so this is correlated with the breadth learning outcome. Excellence here means that the assumptions, conclusions, and the logic from one to the other are clearly expressed.

There is no precise system for determining marks, but a linear regression reveals that marks will usually be close to $45.5 + 2.7m_1 + 3.8m_2 + 4.4m_3$, where m_i is the mark on learning outcome i on the $0 - 4$ scale. This formula is less accurate at both extremes – all marks between 0 and 100 are possible.

Note that exams have become longer and more difficult in recent years to give multiple opportunities to show the third learning outcome. Thus, it has become easier to get a high mark.

Notes for Advanced Mathematical Economics students. You will be rated as no/almost/ok/good/excellent (i.e. 0 to 4) on the following learning outcomes:

- Formulating a model. Excellence here means the absence of important mistakes.
- Applying theorems to models. Excellence here usually means applying three techniques correctly.

- Proving mathematical theorems. This is by far the hardest learning outcome. Excellence here means proving something difficult (most of the questions in Part B qualify) with clear reasoning.

Earning first class honours (70+) requires demonstrating all three learning outcomes. There is no precise system for determining marks, but a linear regression reveals that marks will usually be close to $49 + 2.1m_1 + 2.5m_2 + 5.6m_3$, where m_i is the mark on learning outcome i on the $0 - 4$ scale. This formula is less accurate at both extremes – all marks between 0 and 100 are possible. I strongly recommend that students attempt Part A first.

Questions 20, 21, 24, and 27 are specifically written for Advanced Mathematical Economics. All other questions are from Microeconomics 1, which covers different material. Many questions are based on material that was taught, but only in passing. These questions would be examinable as more advanced questions (and hence would attract a bigger reward if correctly answered.) These questions are *not* examinable, as they are based on ideas that were not covered in lectures at all:

1 (v), (vi), (vii).

2 all examinable.

3 (ii), (vii), (viii).

4 (iv), (vi), (vii).

5 (vi).

6 (ii), (vi), (vii).

7 (ii), (vi).

8 (ii), (iv).

9 (v).

10 (ii), (v), (vi).

11 (vi).

12 (iii), (v).

13 (ii), (vi), (vii).

14 (vii).

15 (ii), (v), (vi).

16 (iv), (vi).

17 (ii), (vi).

18 (ii), (vi), (vii).

19 (vi).

20 –

21 –

22 (iv), (v), (vi).

23 (ii), (v), (vi), (vii).

24 –

25 (ii), (iv), (v).

26 (iv), (v), (vi).

27 –

28 (ii), (vi).

Advice for Answering Exam Questions

Generic Advice.

- There is no need to add extra complications into the model. For example, if the question does not mention time, then there is no need to put multiple time periods into the model.
- If you can't figure out the answer, don't pretend you know it. It's better to explain what you are confused about – a well written statement of confusion can illustrate that you know the material very well, and give you a very good mark.
- Even if you misformulate your model, this shouldn't stop you from answering subsequent parts. But if the model then seems inconsistent with the question (e.g. the question asks "show real wages are higher" when in your model, this is not true) then please do not try to prove the impossible. Instead, please either explain why the question is inconsistent, or if you're not sure, explain why you are stuck and can't complete your argument.

- Students often incorrectly identify the envelope formula as a first-order condition. It's not. First-order conditions are about optimal choices. If you are differentiating with respect to prices, you are not doing a first-order condition, because in competitive markets, nobody can choose prices.

- Students often confuse value functions and objective functions. For example, students often (mistakenly) write that a firm's first-order conditions with respect to the number of workers involves differentiating the profit function (rather than the firm's objective function) with respect to its labour input. But the profit function is a function of prices, not quantities, so it makes no sense to differentiate it with respect to a quantity.

- You can introduce assumptions at any point in the paper. For example, if you discover in part (iv) that you need to assume that the production function is concave, then you can write that assumption in your answer to part (iv). You do not need to revise your answer to part (i).

- In proofs and calculations, please write with complete grammatical sentences, including punctuation.

- In proofs, be careful to distinguish between "there exists" and "for all".

- In proofs, be careful to distinguish between set membership ($\in$) and subsets ($\subseteq$).

A Basic Checklist. A mark below 50% means something important was missing from your model. For example, you might have had two different markets with the same price, or a firm buying something (like a wholesale good) without using it in production. Here is a check-list of important ingredients of every economic model:

- Any notation is fine, but you must define it.

- When writing down the agents' optimisation problems, you should always write the choice variables under the *max*.

- In competitive models, agents only choose quantities, not prices.

- Every market has one (and only one) price. For example, labour markets have only one price if all types of labour are equally valued (by buyers and sellers). On the other hand, if workers have preferences over their profession, or firms value some workers above others, then these are separate markets and have separate prices.

- Every cost should also have a corresponding benefit (and vice versa). There are exceptions to this rule (e.g. inelastic labour supply), but think carefully about this.
- Every market should have a market-clearing condition. Thus, there are always an equal number of prices and market-clearing equations. It also means you need to define notation for both supply and demand. (In the sample solutions, I typically write firm decisions in upper case, and household decisions in lower case.)

Notation: Notation for partial derivatives: there are many common (correct) ways to write partial derivatives, including

$$\frac{\partial}{\partial x} f(x, y) \tag{1}$$

$$\frac{\partial f(x, y)}{\partial x} \tag{2}$$

$$f_x(x, y) \tag{3}$$

$$f_1(x, y) \tag{4}$$

$$D_x f(x, y) \tag{5}$$

$$D_1 f(x, y) \tag{6}$$

$$\nabla_x f(x, y) \tag{7}$$

$$\nabla_1 f(x, y). \tag{8}$$

Writing

$$f'_x(x, y) \tag{9}$$

is *not* standard, so I suggest you avoid it. (It is unambiguous though, so it wouldn't lose you marks in my exams.)

The notation $f'(x, y)$ or $Df(x, y)$ or $\nabla f(x, y)$ does *not* represent a partial derivative, but rather the *total* derivative, i.e. the vector (or matrix) of partial derivatives of f. Please don't write this if you mean a partial derivative.

Question 1. Consider a pure-exchange economy in which all goods are produced from oil by home production over 2 time periods. Only oil is traded. There are two households and two oil deposit sites of size 1. The first site is owned by household A, and oil can be extracted from it at any rate over the 2 periods. The second site is owned by household B, but oil production is only possible in the second period. Both households have the same preferences, which are impatient discounted utility with the same per-period utility function which is strictly concave.

(i) Define an equilibrium in this economy.

Answer: **Household A:** consumption c_1^A, c_2^A in periods 1 and 2, oil sales k_1^A, k_2^A, utility u, discount factor β, prices p_1, p_2,

$$\begin{aligned} &\max_{c_1^A, c_2^A, k_1^A, k_2^A \geq 0} \quad u(c_1^A) + \beta u(c_2^A) \\ &\text{s.t.} \quad p_1 c_1^A + p_2 c_2^A = p_1 k_1^A + p_2 k_2^A \\ &\qquad\quad k_1^A + k_2^A = 1. \end{aligned}$$

Household B.

$$\begin{aligned} &\max_{c_1^B, c_2^B \geq 0} u(c_1^B) + \beta u(c_2^B) \\ &\text{s.t. } p_1 c_1^B + p_2 c_2^B = p_2 \cdot 1 \end{aligned}$$

Market clearing.

$$\begin{aligned} c_1^A + c_1^B &= k_1^A \\ c_2^A + c_2^B &= k_2^A + 1. \end{aligned}$$

Equilibrium. An equilibrium is a vector of quantities $c_1^{*A}, c_2^{*A}, c_1^{*B}, c_2^{*B}, k_1^{*A}, k_2^{*A}$ and prices p_1^*, p_2^* such that the quantities solve the households' problems above, and the markets clear.

(ii) Write down the egalitarian social planner's problem (i.e. assuming that the social planner puts equal weight on the households.) What allocation would she choose?

Answer:

$$\begin{aligned} &\max_{c_1^A, c_2^A, c_1^B, c_2^B \geq 0} u(c_1^A) + u(c_1^B) + \beta u(c_2^A) + \beta u(c_2^B) \\ &\text{s.t.} \quad c_1^A + c_1^B \leq 1 \\ &\qquad\quad c_1^A + c_1^B + c_2^A + c_2^B \leq 2. \end{aligned}$$

The social welfare function is strictly concave, so there is a unique optimal allocation. By inspection, the first-order conditions for the two households are identical, so the solution gives both households the same consumption paths. Thus, the social planner's problem reduces to:

$$\begin{aligned}
&\max_{c_1, c_2 \geq 0} \; 2u(c_1) + \beta 2u(c_2) \\
&\text{s.t.} \quad 2c_1 \leq 1 \\
&\qquad\quad 2c_1 + 2c_2 \leq 2.
\end{aligned}$$

Does the first constraint bind? To check, we will solve without it. The FOCs w.r.t. c_1 and c_2 without the constraint is:

$$2u'(c_1) = \lambda 2 \quad \text{and} \quad 2\beta u'(c_2) = \lambda 2.$$

Hence, $u'(c_1) = \lambda < \lambda/\beta = u'(c_2)$, which means that $u'(c_1) < u'(c_2)$. Since u is concave, u' is decreasing, so $c_1 > c_2$. Thus, the constraint is violated if we drop it. We conclude that it binds, which means that $c_1 = 0.5$ and $c_2 = 0.5$.

Summary: because the social planner values both households equally, and the households have strictly concave utility, both households follow equal consumption paths. The households are impatient, so the social planner is tempted to give them more consumption in the first period than the second. However, this is infeasible, because there is not enough oil in the first period. Thus, the households have equal consumption across time.

(iii) In equilibrium, how do oil prices change over time?

Answer: The FOCs for households A are

$$u'(c_1^A) = \lambda^A p_1 \quad \text{and} \quad \beta u'(c_2^A) = \lambda^A p_2.$$

Dividing the top by the bottom gives

$$\frac{u'(c_1^A)}{u'(c_2^A)} = \beta \frac{p_1}{p_2},$$

or equivalently,

$$p_2 = \frac{u'(c_2^A)}{u'(c_1^A)} \beta p_1.$$

A similar procedure gives

$$p_2 = \frac{u'(c_2^B)}{u'(c_1^B)}\beta p_1.$$

Since aggregate consumption can not be bigger in period 1, one (and hence both) of the fractions must be less than one. Hence $p_1 > p_2$.

(iv) In equilibrium, which household is better off? Explain.

Answer: Since prices are decreasing over time, the first household's endowment is worth more.

(v) Suppose there is a bubble, in the sense that in the last period, oil prices are too high and there is excess supply of oil in the last period. What would happen in the first period? (Hint: Walras' law.)

Answer: Walras' law applies. (The version in class was only for pure-exchange economies; oil storage can easily be accommodated with home production.) Walras' law says that there must be excess demand in some other market. Since there are only two markets, it must be excess demand for oil in the first period.

(vi) * Which assumptions above about the households' utility are relevant for Debreu's theorem about additively separable preferences? Which assumptions go beyond the conclusion of Debreu's theorem?

Answer: The question assumes the conclusion of Debreu's theorem (and more), that preferences are additively separable. Debreu requires there to be at least three time periods, and for preferences to be additively separable. The assumptions of discounted utility and impatience are additional assumptions made by the model.

(vii) * What additional assumptions are needed to ensure existence of equilibria in this economy?

Answer: None. The utility functions are strictly quasi-concave, so the excess demand function is continuous. The choice spaces are convex and compact, so the proof of the existence theorem would not have any serious obstacles.

Question 2. The cashew tree is native to the Amazon forest in Brazil, its fruit is about the same size as an apple. The juice of the flesh of the fruit is popular in Brazil (along with açaí, acerola, guava, mango, papaya, and many others... but ignore those!) Each fruit also contains a single seed, which when toasted becomes the cashew nut which is popular all over the world.

(i) The firm chooses how many cashew fruits to grow (which requires labour), and then sells the juice and nuts. Assume that no work is required to extract the juice and nuts – only growing requires labour. Write down the firm's profit function.

Answer: Notation: J juice, N nuts, L labour, $f(L)$ fruit production function, p^J, p^N, p^L prices

$$\pi(p^J, p^N, p^L) = \max_L (p^J + p^N) f(L) - Lp^L.$$

(ii) Write down the firm's cost function. Hint: you will need two quantities in the state variable (as well as factor prices).

Answer: Notation: Y^J, Y^N are production targets,

$$\begin{aligned} C(Y^J, Y^N, p^L) = &\min_L Lp^L \\ &\text{s.t. } f(L) \geq Y^J \text{ and } f(L) \geq Y^N. \end{aligned}$$

(iii) There are several identical households that supply labour and consume cashews and cashew juice and hold equal shares in the cashew firm. Write down a general equilibrium model of the economy.

Answer: Focus on symmetric equilibria, in which all households make the same decisions.

Households: H households, $\Pi = \pi(p^J, p^N, p^L)$ aggregate profits,

$$\begin{aligned} &\max_{c^J, c^N, l} u(c^J, c^N, 1 - l) \\ &\text{s.t. } p^J c^J + p^N c^N = p^L l + \Pi/H. \end{aligned}$$

Firms: As above.

Equilibrium: Prices (p^{*J}, p^{*N}, p^{*L}) and quantities (c^{*J}, c^{*N}, l^*) for households and (Y^{*J}, Y^{*N}, L^*) for the firm such that:

- the quantity decisions are optimal given prices (see above), and
- all markets clear:

$$\begin{aligned} Hc^{*J} &= Y^{*J} \\ Hc^{*N} &= Y^{*N} \\ Hl^* &= L^*. \end{aligned}$$

(iv) Write down a utility function for the households consistent with the idea that households enjoy cashew nuts more than cashew juice. What can you say about equilibrium prices in this case?

Answer: For example, pick $u(c^J, c^N, r) = \log c^J + 2\log c^N + r$, where r is relaxation time. The FOCs for c^J and c^N are

$$\begin{aligned} \frac{1}{c^J} &= \lambda p^J \\ \frac{2}{c^N} &= \lambda p^N. \end{aligned}$$

From the firm's production technology and market clearing, we know that $c^J = c^N$ in all (symmetric) equilibria. Dividing the second FOC by the first and rearranging gives

$$p^N = 2p^J,$$

i.e. cashew nuts are twice as expensive in this example. Even though the social cost of producing nuts is the same as juice, the marginal social opportunity cost of consuming a nut is higher, because it deprives the other households of something more valuable.

(v) Does the firm have increasing marginal cost in both products?

Answer: Yes, the cost function

$$\begin{aligned} C(Y^J, Y^N; w) = &\min_{L \geq 0} wL \\ &\text{s.t. } f(L) \geq \max\left\{Y^J, Y^N\right\} \end{aligned}$$

is convex in the output targets. If the cheapest way to produce $Y = (Y^J, Y^N)$ is L units of labour, and to produce $\hat{Y} = (\hat{Y}^J, \hat{Y}^N)$ is $\hat{L}$ units, then we just need to check that producing $aY + (1-a)\hat{Y}$ output requires at most $\alpha L + (1-\alpha)\hat{L}$ labour. To see this, $C(Y; w) = wL$ and $C(\hat{Y}; w) = w\hat{L}$, and if $\alpha L + (1-\alpha)\hat{L}$ meets the production targets $\alpha Y + (1-\alpha)\hat{Y}$, then

$$C(\alpha Y + (1-\alpha)\hat{Y}; w) \leq w[\alpha L + (1-\alpha)\hat{L}] = \alpha C(Y; w) + (1-\alpha)C(\hat{Y}; w).$$

Looking at the juice, we know that

$$f(L) \geq Y^J \tag{10}$$
$$f(\hat{L}) \geq \hat{Y}^J, \tag{11}$$

because L and $\hat{L}$ labour generates at least these amounts of juice. By the concavity of the production function f, we know that $f(\alpha L + (1-\alpha)\hat{L}) \geq \alpha f(L) + (1-\alpha) f(\hat{L})$. Thus, taking the convex combination of the equations (10) and (11) and combining with this convex inequality gives

$$f(\alpha L + (1-\alpha)\hat{L}) \geq \alpha Y^J + (1-\alpha)\hat{Y}^J.$$

That is, the intermediate amount of labour produces at least the intermediate amount of juice. A similar line of reasoning applies to the nuts.

(vi) Sketch a graph of the firm's marginal cost of producing cashew juice, holding fixed the number of cashew nuts being produced at 3.

Question 3. (Micro 1 class exam in December 2012) A farm produces food from labour. However, the farm does not have a distribution network, so it can not sell the food directly to the households. Rather, it must sell the food to a supermarket at a wholesale price, which then resells to households at a retail price. The supermarket buys food and labour, which it uses to resell the food. Some food might get wasted; more labour means less food gets wasted. All households are identical, and supply labour to both firms.

(i) Formulate an economy by writing down the households' and firms' value functions, and the market clearing conditions. Focus attention on symmetric equilibria, i.e. in which all households make the same decisions. (Hint: you might find it helpful to consider the wholesale food a completely separate good. Don't forget profits.)

Answer: **Household.** p retail food price, w wage, c consumption, l labour, H number of households, $u(c, l)$ utility function, $\Pi = \Pi^F + \Pi^S$ firms' profits, value

$$v(p, w) = \max_{c,l} u(c, l)$$
$$\text{s.t. } pc = wl + \frac{\Pi}{H}.$$

Farm. D_F wholesale good produced, $D_F = f(L_F)$ production function, ϕ wholesale price, value

$$\pi^F(\phi, w) = \max_{L_F} \phi f(L_F) - wL_F.$$

Supermarket. D_S wholesale good purchased, C_S retail food sold, $C_S = g(L_S, D_S)$ production function, value

$$\pi^S(p, \phi, w) = \max_{L_S, D_S} pg(L_S, D_S) - \phi D_S - wL_S.$$

Equilibrium. A symmetric allocation consists of quantities for households (c^*, l^*), the farm (D_F^*, L_F^*), and the supermarket (C_S^*, D_S^*, L_S^*). These choices, along with prices (p^*, ϕ^*, w^*) and profits (Π^{F*}, Π^{S^*}) form an equilibrium if the

- choices solve the problems defined above,
- profits match: $\Pi^{S*} = \pi^S(p^*, \phi^*, w^*)$ and $\Pi^{F^*} = \pi^F(\phi^*, w^*)$.
- food clears: $Hc^* = C_S^*$.

- wholesale clears: $D_S^* = D_F^*$.
- labour clears: $Hl^* = L_S^* + L_F^*$.

(ii) Select a constraint which may be dropped by Walras' law.

Answer: Eg: "food clears."

(iii) Suggest how an endogenous variable may be eliminated, since inflation of all prices by an equal factor does not affect decisions.

Answer: Eg: set $w^* = 1$.

(iv) Show that the supermarket's profit function is convex. (Hint, you may use the following theorem from class: Suppose V is the upper envelope of convex functions, i.e. $V(a) = \max_b v(a, b)$ where $v(\cdot, b)$ is a convex function for each b. Then V is convex.)

Answer: To apply the theorem, the choice variable b corresponds to the quantities (D_S, L_S), the state variable a corresponds to prices (p, ϕ, w), and the function $v(a, b)$ corresponds to $pg(L_S, D_S) - \phi D_S - wL_S$, which is linear in prices. Since linear functions are convex, the theorem implies that the upper envelope, $\pi^S(p, \phi, w)$ is convex.

(v) Show that the supermarket responds to a wholesale price increase by buying less.

Answer: By the envelope theorem,

$$\frac{\partial \pi^S(p, \phi, w)}{\phi} = \frac{\partial}{\partial \phi} \left[pg(L_S, D_S) - \phi D_S - wL_S\right]_{L_S = L_S(p,\phi,w), D_S = D_S(p,\phi,w)} = -D_S(p, \phi, w).$$

Differentiating and multiplying by -1 on both sides gives

$$-\frac{\partial^2 \pi^S(p, \phi, w)}{\phi^2} = \frac{\partial D_S(p, \phi, w)}{\partial \phi}.$$

Since π^S is convex, the left side is negative. Thus, the right side is negative, so the sales policy is decreasing in the wholesale price ϕ.

(vi) There have been protests recently that the (equilibrium) retail price is much higher than the wholesale price, which the households feel is grossly unfair. They propose introducing a profit tax of 50% to be redistributed equally among households, a price markup ceiling of 10%, and a minimum wage increase of 20%. Would this policy make the households better off (under standard assumptions, like increasing utility functions)?

Answer: No. By the first welfare theorem, the original allocation was efficient. Thus, it is infeasible to make all households better off. In fact, since all households have the same budget constraint and utility function, they all attain the same equilibrium utility, so no household would be better off.

(vii) * Prove that the supermarket's policy is continuous if its production function is strictly concave. You may assume that the supermarket only has space to accommodate a maximum number of workers and amount of food.

Answer: The strict concavity of the firm's objective implies that the optimal policy $\psi(P)$ as a function of the price vector P is unique. Now suppose for the sake of contradiction that a sequences of price vectors P_n converges to P^*, but that $\psi(P_n)$ does not converge to $\psi(P^*)$. Since the number of workers and food are limited, the choice space is compact, so we may assume without loss of generality that $\psi(P_n)$ converges to some point, (L, D). But by continuity of the supermarket's objective, (L, D) and $\psi(P^*)$ give the same profit, which contradicts the uniqueness of the optimal policy.

(viii) * To prove existence of equilibrium using Brouwer's fixed point theorem, it is important that the set of possible prices are compact. Explain why this is important, and how to accommodate this requirement.

Answer: One way to prove existence is to show that there is a fixed point of some price-adjustment function $\phi : P \mapsto P'$. Boundedness of the possible price set is important, as inflation might rule out fixed points (eg: $\phi(P) = P + (1, \ldots, 1)$ has no fixed point.) Closedness is important to rule out a hole at a point that would have been the fixed point. It is straightforward to compactify the price set by normalising prices rescaling them to sum to 1. This is possible, because only relative prices matter – rescaling does not affect incentives.

Question 4. (Micro 1 class exam in December 2012) Sackman, Erickson, and Grant (1968) conducted an experiment on computer programmers, which they published in the Communications of the Association of Computing Machinery. They summarised their findings with the following poem:

> When a programmer is good,
> He is very, very good,
> But when he is bad,
> He is horrid.

Even though the programmers were quite experienced, there was very wide disparity in their abilities. They found the best programmer writes their code about 20 times more quickly than the worst programmer. They debug it 28 times more quickly, the final code runs about 10 times faster, and so on. Follow-up studies report similar disparities, and it has become conventional wisdom that the best computer programmers are about 10 times more productive than the median.

Suppose there is a mediocre and a brilliant computer programmer. Assume that one hour of work by the brilliant programmer is a perfect substitute for ten hours of work by the mediocre programmer. The households are otherwise identical and hold equal shares in the firm.

(i) Write down a model of this economy, and define a general equilibrium for it.

Answer: **Firm.** Wages w_m and w_b, hours L_m and L_b sale price p, production function f. Profit

$$\pi(p, w_m, w_b) = \max_{L_m, L_b} pf(L_m + 10L_b) - w_m L_m - w_b L_b.$$

Households. Household $h \in \{m, b\}$ chooses consumption c_h and hours l_h to solve

$$\begin{aligned} &\max_{c_h, l_h} u(c_h, l_h) \\ &\text{s.t. } pc_h = w_h l_h + \frac{\Pi}{2}, \end{aligned}$$

where Π is the equilibrium firm profit.

Equilibrium. An equilibrium consists of prices (p^*, w_m^*, w_b^*) and quantities $(c_b^*, c_m^*, l_b^*, l_m^*, L_b^*, L_m^*)$ such that:

- Each decision maker (the households, and the firms) find these quantity choices optimal given prices – see above.
- Consumption clears: $c_m^* + c_b^* = f(L_m^* + 10L_b^*)$.

- The labour markets clear: $L_b^* = l_b^*$ and $L_m^* = l_m^*$.

(ii) Show that in every equilibrium in which both programmers are hired, the brilliant programmer's wage is ten times higher than the mediocre programmer's wage.

Answer: The firm's FOCs wrt L_b and L_m are, respectively

$$10pf'(L_m + 10L_b) = w_b \quad \text{and} \quad pf'(L_m + 10L_b) = w_m.$$

Dividing the first by the second gives

$$10 = w_b/w_m.$$

(If a worker isn't hired, then we would need a Lagrange multiplier for the constraint of non-negative hours.)

This maths is simply saying: since one hour of brilliant time is a perfect substitute for ten hours of mediocre time, these two options should cost the same. Otherwise, the firm would go for the cheaper option.

(iii) Show that in every equilibrium, the brilliant programmer is better off than the mediocre programmer.

Answer: The brilliant programmer could make the same choice as the mediocre programmer, and still have money left over to buy more.

(iv) Depending on the preferences of the households, the brilliant programmer might work longer or shorter hours. Draw the indifference curves in a way that indicates the brilliant programmer working *less* than the mediocre programmer.

(v) Some people think that the problem is that mediocre programmers are lazy, and they just need some extra incentives to work hard. In the context of your model, would giving the programmers stock options, 100% bonus pay upon project completion and hiring a masseuse and celebrity chef make everyone better off?

Answer: No. By the first welfare theorem, the equilibrium is efficient. Under the feasibility assumptions of the model, there is no allocation that makes everybody better off.

(vi) The mediocre programmer has another more Machiavellian proposal for increasing productivity. He proposes asking the government to issue a large lump-sum tax on the brilliant programmer, which will force her to work long hours to repay her (government-imposed) debt. The mediocre programmer further proposes the he receive the taxes. Would this proposal work?

Answer: Yes. An allocation in which the mediocre programmer has high consumption supported by the brilliant programmer working very hard is efficient (albeit "unfair"). Thus, by the second welfare theorem, there exist lump-sum taxes to implement this allocation as an equilibrium.

(vii) * Discuss the problems with proving existence in this economy.

Answer: The firm has a bang-bang solution to hiring workers. If brilliant programmer's wage is not exactly 10 times the mediocre programmer's wage, then the firm will specialise in hiring one of them. Thus, the firm's policy is discontinuous, which is an obstacle to applying Brouwer's fixed point theorem.

Question 5. (Micro 1 degree exam in May 2013) We eat about 300 billion apples every year, but most of these apples can not be eaten directly from the tree. The problem is that apples only ripen in Autumn, and apples consumed at other times must be stored. On the other hand, lettuce may be grown in all seasons, so it is never necessary to store it. Henceforth, assume it is non-storable.

Suppose there are just two seasons (Autumn and Spring) and two foods (lettuces and apples). Farmers are endowed with apples in Autumn, and lettuce in equal quantities in both Autumn and Spring. There is a storage firm (owned by the farmers) that can refrigerate apples until the Spring. The storage technology does not require any labour or other resources to operate. However, as they store more fruit, they become less effective and an increasing fraction of apples go bad.

(i) Define a general equilibrium in this setting, focusing attention on symmetric equilibria in which all farmers make the same decisions as each other.

Answer: **Farmers:** there are H of them, time $t \in \{1, 2\}$, apple endowment e^A, lettuce endowment e^L, utility function u, apple prices p_1^A, p_2^A and lettuce prices p_1^L, p_2^L, apple consumption a_1, a_2 and lettuce consumption l_1, l_2, firm profit π.

$$\begin{aligned}&\max_{a_1,a_2,l_1,l_2} u(a_1, a_2, l_1, l_2)\\ &\text{s.t.} \quad p_1^A a_1 + p_2^A a_2 + p_1^L l_1 + p_2^L l_2 = p_1^A e^A + (p_1^L + p_2^L) e^L + \pi/H.\end{aligned}$$

Storage firm: A_1 apples put into storage, $A_2 = f(A_1)$ applies taken out of storage

$$\pi(p_1^A, p_2^A) = \max_{A_1} p_2^A f(A_1) - p_1^A A_1.$$

Market clearing:

$$\begin{aligned}Ha_1 + A_1 &= He^A\\ Ha_2 &= A_2\\ Hl_1 &= He^L\\ Hl_2 &= He^L.\end{aligned}$$

(ii) Is it possible to normalise apples prices to 1?

Answer: No, it's only possible to normalise one price, e.g. $p_1^A = 1$.

(iii) Show that if the storage technology is perfect, then apples prices are equal in both seasons.

Answer: Storage firm's first-order conditions:

$$p_2^A f'(A_1) = p_1^A.$$

(This first-order condition holds in any equilibrium in which $a_2 > 0$.) Since $f' = 1$, we conclude that $p_2^A = p_1^A$.

Comment. The intuition behind this mathematics is as follows: If $p_1^A < p_2^A$ then the firm would try to store an infinite amount of apples (so there would be no optimal choice for the firm). If $p_1^A > p_2^A$, then the firm would store nothing.

(iv) Show if the storage technology involves some spoilage, that apples are more expensive in Spring than Autumn.

Answer: Look at the storage firm's first-order condition (see above). Since there is some wastage, $f'(A_1) < 1$, which means that $p_2^A > p_1^A$.

(v) Suppose that the farmers' preferences have a discounted utility representation. (i.e. Time separable preferences that can be written in an additively separable fashion, with per-period utility functions being identical.) Moreover, assume that the farmers have decreasing marginal utility in apple and lettuce consumption. (a) Write the farmers' first-order conditions, (b) show that the farmers consume more apples in Spring than Autumn, and (c) write the farmer's problem using a Bellman equation.

Answer: Discounted utility representation:

$$u(a_1, l_1) + \beta u(a_2, l_2).$$

(i) Farmers' first-order conditions:

$$\begin{aligned} u_1(a_1, l_1) &= \lambda p_1^A \\ u_2(a_1, l_1) &= \lambda p_1^L \\ \beta u_1(a_2, l_2) &= \lambda p_2^A \\ \beta u_2(a_2, l_2) &= \lambda p_2^L. \end{aligned}$$

(ii) By market clearing and symmetry, we know that $l_1 = l_2$. Therefore, we have that

$$\lambda = \frac{u_1(a_1, l_1)}{p_1^A} = \frac{\beta u_1(a_2, l_1)}{p_2^A}.$$

Since $p_2^A > p_1^A$ (see the previous question), we deduce that

$$u_1(a_1, l_1) < u_1(a_2, l_1).$$

Since $u_1(\cdot, l_1)$ is decreasing due to decreasing marginal utility, we conclude that $a_2 < a_1$.

(iii) Let m be money saved for the second period. Bellman equation:

$$\max_{a_1,l_1,m} u(a_1, l_1) + \beta V(m)$$
$$\text{s.t. } p_1^A a_1 + p_1^L l_1 + m = p_1^A e^A + p_1^L e^L,$$

where the second period value function is

$$V(m) = \max_{a_2,l_2} u(a_2, l_2)$$
$$\text{s.t. } p_2^A a_2 + p_2^L l_2 = m + p_2^L e^L.$$

(vi) Now suppose that one farmer is extra productive, and has double the endowments of all of the other farmers. The other farmers have a smaller endowment so that the aggregate endowments are identical. Think about the prices in the following scenarios:

(a) The original symmetric equilibrium.

(b) The new equilibrium (with the extra productive farmer).

(c) A new equilibrium (with the extra productive farmer) in which the productive farmer is taxed so that the equilibrium allocation is the same as in (a).

Do any of these scenarios share the same equilibrium prices?

Answer: Yes, scenarios (a) and (c) by the Second Welfare Theorem.

(vii) Show that the farmers' second-period value function is concave and ** differentiable.

Answer: First, V is concave. Suppose a_2', l_2' are optimal choices at m', and a_2'', l_2'' are optimal choices at m''. Then for any $t \in [0, 1]$,

$$\begin{aligned}
&V(tm' + (1-t)m'') \\
&\geq u(ta_2' + (1-t)a_2'', tl_2' + (1-t)l_2'') \\
&\geq tu(a_2', l_2') + (1-t)u(a_2'', l_2'') \\
&= tV(m') + (1-t)V(m'').
\end{aligned}$$

Second, by the Benveniste-Scheinkman theorem, V is differentiable at $m > 0$.

Question 6. (Micro 1 degree exam in May 2013) Suppose there are two countries of equal population. However, the big country has twice the amount of land, so that each household located there has twice the land endowment of households in the small country. Each country has an agricultural firm that transforms labour and land into food. Food can be traded on the international market. However, labour and land are more complicated. Each firm is owned equally by the citizens of its own country, and can only grow food on its own country's land. We say that workers migrate if they work for the other country's firm, although we assume that migration is costless.

(i) Write down a general equilibrium model of the labour, food and land markets. (Hint: treat labour and food as unified international markets, but land as national markets.)

Answer: **Households:** from country $i \in \{0, 1\}$ where 1 is big and 0 is small, food consumption x_i, food price p, labour supplied h_i, wages w, land rental price r_i, land endowment e_i, profit of own country's firm π^i, utility function u, number of households in each country N,

$$\begin{aligned} &\max_{x_i, h_i} u(x_i, h_i) \\ &\text{s.t.} \quad px_i = wh_i + r_i e_i + \pi^i / N. \end{aligned}$$

Firms: land rented by firm i is L_i, labour hired H_i, food produced $X_i = f(L_i, H_i)$.

$$\pi^i(p, w, r_i) = \max_{L_i, H_i} pf(L_i, H_i) - wH_i - r_i L_i.$$

Market clearing:

$$\begin{aligned} Ne_0 &= L_0 \\ Ne_1 &= L_1 \\ Nh_0 + Nh_1 &= H_0 + H_1 \\ Nx_0 + Nx_1 &= X_0 + X_1. \end{aligned}$$

Equilibrium. An equilibrium is a vector of quantities $(x_0^*, x_1^*, h_0^*, h_1^*, X_0^*, X_1^*, H_0^*, H_1^*, L_0^*, L_1^*)$ and prices r_0^*, r_1^*, w^*, p^* such that the quantities solve the households' and firms' problems above.

(ii) Suppose that at some (out-of-equilibrium) prices, the food and labour markets clear, but there is excess demand of the small country's land. What does Walras' law say about the market for the large country's land?

Answer: Walras' law says that if there is excess demand in one market, then there is excess supply in another market. By process of elimination, there must be excess supply of land in the large country at these prices.

(iii) Show that the small country's firm's profit function is convex in prices.

Answer: The profit function is the upper envelope of linear functions, (one function for each input choice). Therefore it is convex.

(iv) Show that if wages increase, the small country decreases its demand for labour.

Answer: By the envelope theorem,

$$\frac{\partial \pi^0(p, w, r_0)}{\partial w} = -H_0(p, w, r_0).$$

Since the profit function is convex, both sides of this equation are increasing in w. We conclude that labour demand decreases when wages increase.

(v) Show that if the production technology has constant returns to scale, and leisure is a normal good, then there is some migration from the small to the big country. (Hint: functions that are homogeneous of degree 1, i.e. satisfy the property that $f(tx, ty) = tf(x, y)$, also have the property that $f_x(2x, 2y) = f_x(x, y)$ for all (x, y).)

Answer: The firms' labour first-order conditions are:

$$pf_H(L_0, H_0) = w$$
$$pf_H(L_1, H_1) = w.$$

Constant returns to scale implies that f is homogeneous of degree 1. Since $L_1 = 2L_0$, it follows that

$$f_H(L_1, H_1) = f_H(L_0, H_1/2).$$

The ratio of the labour first-order conditions becomes

$$1 = \frac{f_H(L_0, H_0)}{f_H(L_1, H_1)} = \frac{f_H(L_0, H_0)}{f_H(L_0, H_1/2)},$$

which implies that $H_1 = 2H_0$, i.e. the big country's firm hires twice as many worker hours as the small country's firm.

The firms' land first-order conditions are

$$pf_L(L_0, H_0) = r_0$$
$$pf_L(L_1, H_1) = r_1.$$

Since $(L_1, H_1) = 2(L_0, H_0)$, we deduce that $r_0 = r_1$. This means that the workers in the big country have more non-labour income (land prices are the same but endowments bigger, and profits are bigger in the big country's firm), so they work less as leisure is a normal good. It follows that there is net migration from the small to the big country.

(vi) Suppose the two countries plan to federalise into a free-trade zone (like the EU). They are worried about social tensions arising from the inequality of the people from the two countries. Devise a lump-sum tax scheme that creates perfect equality.

Answer: The target allocation (of perfect equality) is efficient, so the Second Welfare Theorem implies that lump sum taxes may implement this allocation. Moreover, the theorem describes the transfers needed. Citizens of each country are given a transfer that is equal to the the market value of their equilibrium consumption (i.e. with perfect equality) less the market value of their endowment. This difference is negative for citizens of the big country.

(vii) * Suppose that households are constrained to work in one country only (of their choice). Discuss how this possibility impedes application of the Brouwer's fixed point theorem to establish existence of equilibria.

Answer: The households no longer have a choice from a convex subset of $\mathbb{R}^n$, because they have a discrete choice about which country to live in. This isn't necessarily a serious problem, however, since Brouwer's fixed point theorem is typically applied in price space, not consumption space. It might make it difficult to prove continuity of the policy functions though (eg: Berge's theorem of the maximum no longer applies.)

Question 7. US comedian Lewis Black has the following to say about solar energy:

> If you ask your congressman why, he'll say "Because it's hard. It's really hard. Makes me want to go poopie." You know why we don't have solar energy? It's because the sun goes away each day, and it doesn't tell us where it's going!

Two countries are endowed with some electricity during the day time. However, they are located on opposite sides of the world, so when it is day time in one country, it is night time in the other. Electricity is non-storable, so the only way to consume electricity at night is to import electricity from the other country. A portion of the electricity is lost in transportation; the fraction lost increases as the amount of electricity transported increases.

Apart from this, the countries are identical: there is one household in each country, they share the same preferences and endowments, and the household in each country owns its own electricity exporter. You may assume preferences are additively separable across time, and they value electricity consumption equally during the day and night with decreasing marginal utility.

(i) Write down a general equilibrium model of this economy for one 24-hour period consisting of one night and day in each country. (Hint: treat electricity in different countries and different times as separate markets.)

Answer: **Households:** country $i \in \{A, B\}$, time $t \in \{1, 2\}$, electricity consumption c_t^i, electricity endowment e_t^i, utility function u, local electricity price p_t^i

$$\begin{aligned} &\max_{c_1^i, c_2^i} u(c_1^i) + u(c_2^i) \\ &\text{s.t.} \quad p_1^i c_1^i + p_2^i c_2^i = p_1^i e_1^i + p_2^i e_2^i + \pi^i. \end{aligned}$$

The question imposes the assumptions that $e_1^B = 0$ and $e_2^A = 0$ and $e_1^A = e_2^B$.

Exporter from country A**:** x_t^A electricity exported from country A in time t, $y_t^B = f(x_t^A)$ electricity imported into country B in time t,

$$\pi^A(p_1^A, p_1^B) = \max_{x_1^A} p_1^B f(x_1^A) - p_1^A x_1^A$$

Exporter from country B**:**

$$\pi^2(p_2^A, p_2^B) = \max_{x_2^B} p_2^A f(x_2^B) - p_2^B x_2^B$$

Market clearing.

$$\begin{aligned} c_1^A + x_1^A &= e_1^A \\ c_1^B &= y_1^B \\ c_2^B + x_2^B &= e_2^B \\ c_2^A &= y_2^A. \end{aligned}$$

Equilibrium. An equilibrium is a vector of quantities $c_1^{*A}, c_2^{*A}, c_1^{*B}, c_2^{*B}, x_1^{*A}, x_2^{*B}, y_1^{*B}, y_2^{*A}$ and prices $p_1^{*A}, p_2^{*A}, p_1^{*B}, p_2^{*B}$ such that the quantities solve the households' and exporters' problems above, and the markets clear.

(ii) It is possible to eliminate equilibrium variables and conditions using (i) price normalisation and (ii) Walras' law. Provide specific examples of how each of these may be done in the context of your model.

Answer: We may (i) normalise $p_2^B = 1$, and (ii) drop the market clearing constraint

$$c_2^A = y_2^A.$$

(iii) Suppose that both distributors discover a perfect transportation technology that prevents any electricity from being lost in transportation. In this case, show that both countries have the same sequence of electricity prices.

Answer: If any electricity is exported, then the first-order conditions for the two distributors apply, and they are:

$$\begin{aligned} p_1^B f'(x_1^A) &= p_1^A \\ p_2^A f'(x_2^B) &= p_2^B. \end{aligned}$$

Since no electricity is lost, $f' = 1$, so we conclude that $p_1^A = p_1^B$ and $p_2^A = p_2^B$.

(iv) Show that if the distributors have a perfect transportation (as above), then the prices are the same. (Hint: look at the households' first-order conditions, and check the market clearing conditions.)

Answer: Since prices are the same in both countries, we write p_1 and p_2. The households' first-order conditions are

$$\begin{aligned} u'(c_1^A) &= \lambda^A p_1 \\ u'(c_2^A) &= \lambda^A p_2 \\ u'(c_1^B) &= \lambda^B p_1 \\ u'(c_2^B) &= \lambda^B p_2, \end{aligned}$$

which imply

$$\frac{p_1}{p_2} = \frac{u'(c_1^A)}{u'(c_2^A)} = \frac{u'(c_1^B)}{u'(c_2^B)}.$$

This means that if $p_1 > p_2$, then both households consume less elecriticity in the first period than the second. But this is infeasible, since the aggregate electricity endowment is equal in both periods.

(v) Consider the proposal of taxing electricity consumption to subsidise electricity distributors to compensate them for the wasted energy lost. Would this proposal make everybody better off?

Answer: No. By the first welfare theorem, every competitive equilibrium is efficient. Therefore, it is not possible to make everybody better off without changing the set of feasible allocations.

(vi) Again, suppose that there is a perfect transportation technology (see above). Consider the proposal of one country to invade the other, and to impose a new lump-sum tax on the victim country's household. The booty is distributed to the invading country's household. Does this make the invading household better off?

Answer: Yes. Applying the first welfare theorem to the old and new equilibria (before and after the invasion), we know that both equilibria are efficient. Before invasion, both households have equal welfare (since they have the same preferences and budget constraint – see above). After invasion, the invading household has higher utility than the invaded, so it must be better off than before (otherwise, this would be Pareto dominated by the before-invasion allocation).

Question 8. (Micro 1 class exam in December 2013) Suppose there are two types of people: words people and numbers people. A medicine factory hires workers into two professions: marketing and engineering. Both types of people can do both types of jobs, but words people are better at marketing, and numbers people are better at engineering. Specifically, one hour of a words person's time spent on marketing is equivalent to two hours of a numbers person's time spent on marketing, and vice versa. Both types of people have the same preferences, and are indiffferent between both professions – they just take the best wage they can find. Everybody knows what type of person they are trading with.

(i) Define an equilibrium for this economy.

Comment: The most common mistake was to assume that wages depended on profession rather than skill. (It's possible to prove that it only depends on skill when the worker has no preference about profession.) It would also be ok to have a different wage for every combination of profession and skill.

Another common mistake was to assume that all words people would be assigned to marketing, and all numbers people to engineering. This depends on the firm's production function – perhaps marketing only plays a minor role in the firm, and the firm needs many people working on engineering – even incompetent workers! Incompetent engineers should get paid less than competent ones, so the wages are not based on profession, but rather on skill in this model. (Wages would depend on both if workers disliked one form of work more than another.) One way to avoid this trap is to think about extreme situations. What if the firm only needs one marketing person? Would the firm still want to hire more words people? If you can't think of a reason why not, then you should accommodate it in the model. Also, part (iii) gave the game away – the allocation problem indicates that how to allocate skills to professions is an important trade-off for the problem. Therefore, it's worth reading the whole question to understand the spirit of it, to make sure you aren't missing something important.

Answer: Workers. Worker type $t \in \{N, W\}$, number of type t workers n_t, medicine consumed m_t, medicine price p, hours of labour supplied h_t, wage w_t, firm profit π, utility $u(h_t, m_t)$.

$$\max_{h_t, m_t} u(h_t, m_t)$$
$$\text{s.t. } p_t m_t = w_t h_t + \frac{\pi}{n_N + n_W}.$$

Factory. Profession $s \in \{E, M\}$, type t worker hours allocated to profession s is written H_{ts}, labour inputs $(H_N, H_W) = (H_{NE} + H_{NM}, H_{WE} + H_{WM})$, medicine produced $M = f(2H_{NE} + H_{WE}, H_{NM} + 2H_{WM})$, profit π given by

$$\max_{H_{NE},H_{WE},H_{NM},H_{WM}} pf(2H_{NE} + H_{WE}, H_{NM} + 2H_{WM}) \\ - w_N(H_{NE} + H_{NM}) - w_W(H_{WE} + H_{WM})$$

Equilibrium. $(p^*, w_W^*, w_N^*, h_W^*, h_N^*, m_W^*, m_N^*, H_{NE}^*, H_{WE}^*, H_{NM}^*, H_{WM}^*, M^*)$ form an equilibrium if the household's and firm's respective choices are optimal as defined above, and the following market clearing conditions are satisfied:

$$\begin{aligned} N_W m_W^* + N_N m_N^* &= M^* \\ N_W h_W^* &= H_W^* \\ N_N h_N^* &= H_N^*. \end{aligned}$$

(ii) Suppose there is excess demand for both types of labour, i.e. at market prices, the firm demands more labour than the workers are willing to supply. Does this mean that there is also excess demand for medicine?

Answer: No. Walras' law implies that there is excess supply of medicine.

(iii) The factory has to make two types of choices: how many workers of each type to hire, and how to allocate them to professions.

(a) Define the firm's output function as the maximum amount of medicine the firm can produce with given labour inputs.

(b) Write down a Bellman equation for the factory relating the firm's cost function to the firm's output function.

(c) Show that the firm's cost function is concave with respect to wages.

(d) Show that if the market wage of numbers people increases, then the firm finds it optimal to meet its production target by hiring fewer numbers people and more words people.

Answer:

(a)

$$\begin{aligned} F(H_N, H_W) = &\max_{H_{NE},H_{WE},H_{NM},H_{WM}} f(2H_{NE} + H_{WE}, H_{NM} + 2H_{WM}) \\ &\text{s.t. } H_{NE} + H_{NM} = H_N \text{ and } H_{WE} + H_{WM} = H_W. \end{aligned}$$

(b)

$$\begin{aligned} c(M; w_N, w_W) = \min_{H_N, H_W} \; & w_N H_N + w_W H_W \\ \text{s.t. } & F(H_N, H_W) = M. \end{aligned}$$

(c) Holding the output target M fixed, the firm's cost function is the lower envelope of a set of linear functions (one function for each feasible pair (H_N, H_W) that can be used to meet the target). The lower envelope of linear functions (which are concave) is concave.

(d) By the envelope theorem,

$$\frac{\partial c(M; w_N, w_W)}{\partial w_N} = H_N(M; w_N, w_W).$$

Since the cost function is concave, the left side is decreasing in numbers wages w_N. It follows that the right side, the number of numbers people hired $H_N(M; w_N, w_W)$ to meet output target M, is a decreasing function of numbers wages w_N. Therefore, more words people must be hired to meet the target.

(iv) Suppose the Words Union has an agreement which guarantees a maximum number of hours for words people only, and that this makes the words people better off. The Numbers Union proposes offering the Words Union a deal: it would tax numbers workers a little bit, and give those taxes to words workers. In return, the Words Union would abandon its maximum hours policy. Is it possible that both unions would agree to this deal?

Answer: Yes. There are three relevant allocations to consider, (i) the competitive equilibrium, (ii) the Words Union allocation, and (iii) the lump-sum tax allocation. By the first welfare theorem, allocation (i) is efficient. Since words people are better off in (ii) than (i), the numbers people must be worse off in (ii) than (i). On the other hand, allocation (ii) need not be efficient. It might be Pareto dominated by another allocation, and hence dominated by an efficient allocation, which we might call (iii). By the second welfare theorem, allocation (iii) can be implemented by lump-sum taxes. Conclusion: if the Numbers Union deal is inefficient, then a deal involving lump-sum taxes to cancel the agreement is Pareto improving, and would be accepted by both unions.

(v) * Prove that the cost function is differentiable with respect to wages.

Answer: We already established that the cost function is concave with respect to wages. Hold the output target M^* fixed, and pick any pair of wages,

(w_N^*, w_W^*). For these wages and output target, there is an optimal hours choice, (H_N^*, H_W^*), and the "lazy" cost function

$$\bar{c}(M^*; w_N, w_N) = H_N^* w_N + H_W^* w_W$$

is a differentiable upper support function for the cost function at (w_N^*, w_W^*). Therefore, by the Benveniste-Scheinkman theorem, the cost function is differentiable at (w_N^*, w_W^*). But the choice of these wages was arbitrary, so the cost function is differentiable everywhere.

Question 9. (Micro 1 class exam in December 2013) A child care centre provides any number of hours of care to several households using two types of labour: babysitters and cleaners. Both types of labour are necessary for production – if either is zero, then no care can be provided. Households can simultaneously supply labour of both types. Households are also endowed with divisible houses, which they can exchange.

Comment: The main difficulties students have with this question are the welfare parts (iv) and (v). The thrust of the question is: do the welfare theorems apply when specialisation is required? You have to know the proofs of the welfare theorems to answer these questions well. The proof of the first welfare theorem does not really require a convex budget constraint (see the sample solution for details), but the second welfare theorem uses it.

(i) Define the concept of a symmetric equilibrium for this economy, in which each household makes the same choice.

Answer: Households. N number of households, b labour on babysitting, c labour on cleaning, w_b wage for babysitting, w_c wage for cleaning, p care price, x child-care services demanded, h housing demand, e housing endowment, q house price, $u(h,x,b,c)$ utility, π firm profits,

$$\max_{h,x,b,c} u(h,x,b,c)$$
$$\text{s.t. } qh + px = qe + w_b b + w_c c + \frac{\pi}{N}.$$

Firm. B baby sitters hired, C cleaners hired, $X = f(B,C)$ care output,

$$\pi(p,w_b,w_c) = \max_{B,C} pf(B,C) - w_b B - w_c C.$$

Equilibrium. $(q^*, p^*, w_b^*, w_c^*, h^*, x^*, b^*, c^*, X^*, B^*, C^*)$ form an equilibrium if the households' and firm's respective choices are optimal, as defined above, and the following market clearing conditions are satisfied:

$$Nh^* = Ne^*$$
$$Nx^* = X^*$$
$$Nb^* = B^*$$
$$Nc^* = C^*.$$

(ii) Suppose at all equilibrium allocations, the households have a higher marginal utility loss of cleaning than babysitting. Show that in every equilibrium, the cleaning wage is higher than the babysitting wage.

Answer: Let λ be the Lagrange multiplier for the budget constraint. The household's first-order conditions with respect to cleaning and babysitting are

$$-\frac{\partial u(h,x,b,c)}{\partial b} = \lambda w_b$$
$$-\frac{\partial u(h,x,b,c)}{\partial c} = \lambda w_c.$$

On the left side, the first line is lower than the second by the assumption. And the right side, it follows that $w_b < w_c$.

(iii) Suppose that the firm's production function is not concave. Does this imply that the profit function is not convex in prices?

Answer: No, it is still convex! The profit function is linear in prices, because it is the upper envelope of linear functions. Specifically for each input vector (B, C), the function

$$g(p, w_b, w_c; B, C) = pf(B, C) - w_b B - w_c C$$

is linear in (p, w_b, w_c), and the profit function is the upper envelope of all g functions.

(iv) Suppose that workers must specialise in at most one profession, babysitting or cleaning. (This isn't a government restriction, just a difficulty of working in these professions.) Are all equilibria efficient? Specifically, is it the case that every equilibrium in this environment is Pareto undominated by every feasible allocation in this environment?

Answer: Yes. The proof of the first welfare theorem is based on the idea that if an allocation Pareto dominates an equilibrium allocation, then that allocation is more valuable at the market prices of the equilibrium allocation, and is therefore infeasible. This proof only applies directly to pure-exchange economies, but can be extended to production economies using the idea of home production. Adding a specialisation constraint would not be a problem for the proof. In particular, it would not affect the key step that at least one household must be unable to afford its consumption in the supposedly Pareto dominating allocation.

(v) * As in the previous part, suppose that workers must specialise in at most one profession, babysitting or cleaning. Can every efficient allocation in this environment be implemented using lump-sum taxes?

Answer: No. First, the proof in class does not apply. It relies on the existence theorem, which is inapplicable since the excess demand function is not continuous: a small change in relative wages could make the household make a discontinuous switch in specialisation. Second, existence is essential – if there is no equilibrium when the endowment equals the efficient allocation, then there will be no way to implement that allocation in a competitive equilibrium with lump-sum taxes.

Question 10. (Micro1 degree exam in May 2014) Suppose there are two rural districts that share an identical agricultural technology for transforming water into food. In the first year, households in both districts are endowed with the same amount of water, which they sell to farms. In the second year, one district suffers a perfectly predictable drought and has no water endowment. Households only directly consume food, and only hold shares in local farms. There are no import/export or migration costs, but food and water are non-storable.

(i) Write down a competitive general equilibrium model of the economy. You may assume households' preferences can be represented by an additively separable utility function.

Comment: Make sure you get your markets right! (There are 4 markets: food and water in periods 1 and 2). It's not a problem if you have extra markets (eg: food in district A in period 1) as long as the logic of your model implies the prices are equal across your artificial markets, and that it is feasible within your economy for food to be reallocated between districts. (Eg: each firm can sell their output to both districts, which would imply the prices are equal – otherwise, firms would specialise in one district).

Make sure you get your choice variables (under the max) right! Many students write that the water endowments were choice variables. I imagine the source of confusion is that students expect the households to have to choose something about water – but get confused when the households didn't consume their water. The most straightforward answer is to assume the households have NO choice – they sell all of their water. Another option is to separately account for the endowment of water and consumption of water, and since the household derives no utility from its consumption, it will sell all of it.

Usually, every cost should have a corresponding benefit (and vice versa). In this question, we have an exception: there is no cost to households of giving up their water endowments. But this makes sense (it was in the question). It's good to double check: have all my costs got benefits?

Answer: Households. Districts $d \in \{A, B\}$ where B suffers the drought, year $t \in \{1, 2\}$, number of households N_d, food consumption c_{dt}, water endowment w_{dt} (the drought makes $w_{B2} = 0$), food price p_t, water price s_t, discount rate β, per-period utility $u(c_{dt})$, farm profit π_d:

$$\max_{c_{d1}, c_{d2}} u(c_{d1}) + \beta u(c_{d2})$$
$$\text{s.t. } p_1 c_{d1} + p_2 c_{d2} = s_1 w_{d1} + s_2 w_{d2} + \frac{\pi_d}{N_d}.$$

Farms. Water demand of farm located in district d is W_{dt}, production function $f(W_{dt})$

$$\pi(p_{d1}, p_{d2}, s_{d1}, s_{d2}) = \max_{W_{d1}, W_{d2}} p_1 f(W_{d1}) + p_2 f(W_{d2}) - s_1 W_{d1} - s_2 W_{d2}.$$

Equilibrium. $(p_t^*, s_t^*, c_{dt}^*, w_{dt}^*, W_{dt}^*)$ form an equilibrium if the households' and firm's respective choices are optimal, as defined above, and the following market clearing conditions are satisfied:

$$\begin{aligned}
N_A^* c_{A1}^* + N_B^* c_{B1}^* &= f(W_{A1}^*) + f(W_{B1}^*) \\
N_A^* c_{A2}^* + N_B^* c_{B2}^* &= f(W_{A2}^*) + f(W_{B2}^*) \\
N_A^* w_{A1}^* + N_B^* w_{B1}^* &= W_{A1}^* + W_{B1}^* \\
N_A^* w_{A2}^* + N_B^* w_{B2}^* &= W_{A2}^* + W_{B2}^*
\end{aligned}$$

(ii) Suppose that some protesters succeed in lowering the price of water in the second period, which leads to excess demand of water in the second period. According to Walras' law, what other consequences would this non-equilibrium behaviour have?

Comment: Students often incorrectly apply Walras' law by identifying a specific market with excess supply.

Answer: If there's excess demand in one market, there must be excess supply in another market. However, Walras' law does not say which market this might occur in.

(iii) Show that each household has a decreasing marginal value of saving for the second year, provided that the household has a decreasing marginal utility of consumption. (Hint: this involves formulating the value of savings.)

Answer: The value of savings m in the second year is

$$V_{d2}(m; p_2, s_2) = u\left(\frac{m + s_2 w_{d2}}{p_2}\right).$$

V_{d2} is a concave function in m, because it is the composition of a concave function u with a linear function. It's derivative, the marginal value of savings, is therefore a decreasing function.

(iv) Show that each household consumes less during the drought.

Answer: The first-order conditions for a household in district d can be simplified to

$$\lambda_d = \frac{u'(c_{d1})}{p_1} = \beta \frac{u'(c_{d2})}{p_2},$$

where λ_d is the Lagrange multiplier for the budget constraint. Since output in the second year, $f(W_2)$ is less than output in the first year, $f(W_1)$, at least one household consumes less in the second year. So that household, in district d, has (by decreasing marginal utility)

$$\frac{u'(c_{d1})}{u'(c_{d2})} < 1.$$

By the first-order condition, the left side of this inequality (the marginal rate of substitution of consumption between the two periods) is the same for all households in equilibrium:

$$\beta \frac{p_1}{p_2} = \frac{u'(c_{d1})}{u'(c_{d2})}.$$

Therefore, all households satisfy the inequality, and hence consume less in the second period.

(v) The government would like to compensate the drought-striken district. Either devise a lump-sum tax policy that would implement smooth (constant) consumption over time for all households, or prove that this task is impossible.

Answer: It is impossible. Any allocation that involves constant consumption over time for all households is *inefficient*, since output is higher in the first period than the second. By the first welfare theorem, any competitive equilibrium is *efficient* (regardless of how endowments are reallocated). Therefore, regardless of the lump-sum taxes chosen, the competitive equilibrium would not involve constant consumption.

(vi) * Write down a function that has the following property: a price vector is a fixed point of that function if and only if there exists an equilibrium with that price vector. Your function should never lead to negative prices. (You may make use of the excess demand function without defining it explicitly.)

Answer: Let $P = (p_1, p_2, s_1, s_2)$ denote a price vector and let $z(P)$ denote the excess demand function. Then the function

$$\phi(P) = \begin{bmatrix} \max\{P_1, P_1 + z_1(P)\} \\ \cdots \\ \max\{P_4, P_4 + z_4(P)\} \end{bmatrix}$$

has the required property. If P has an equilibrium allocation, then $z(P) = 0$ and hence $\phi(P) = P$. Conversely, if P does not have an equilibrium allocation, then by Walras' law, there is excess demand in one market (and excess supply in another market), so $\phi(P) \neq P$.

Question 11. (Micro 1 degree exam in May 2014) Individuals are endowed with one unit of human capital and time. In the first year, individuals divide their time between accumulating human capital (through self-study), labour, and leisure. In the second year, the individuals divide their time between labour and leisure only. A firm produces a consumption good in each year using labour. The contribution of each hour of work to production is proportional to the worker's human capital.

(i) Write down a perfectly competetive model for this market. You may assume the households have additively separable utility, with stationary flow utility. (Hint: the human capital production function should have decreasing marginal product.)

Comment: A common mistake is to have labour and leisure as separate goods. You can split them if you like, but then you should have a time budget constraint.

Answer: Households. Time $t \in \{1, 2\}$, number of households N, consumption c_t, human capital endowent $k = 1$, human capital investment i, human capital production function $g(i)$, labour supply l_t, consumption price p_t, wages w_t, flow utility $u(\cdot, \cdot)$, discount rate β, equilibrium firm profit π. Households solve

$$\max_{c_1,c_2,i,l_1,l_2} u(c_1, l_1 + i) + \beta u(c_2, l_2)$$
$$\text{s.t. } p_1 c_1 + p_2 c_2 = w_1 k l_1 + w_2(k + g(i))l_2 + \frac{\pi}{N}.$$

Firm. Labour demand L_t, production function $f(L_t)$, profit maximisation problem:

$$\pi(p_1, p_2, w_1, w_2) = \max_{L_1,L_2} p_1 f(L_1) + p_2 f(L_2) - w_1 L_1 - w_2 L_2.$$

Equilbrium. $(p_1^*, p_2^*, w_1^*, w_2^*, k^*, i^*, c_1^*, c_2^*, l_1^*, l_2^*, L_1^*, L_2^*)$ forms an equilibrium if the choices solve the household's and firm's problem, and markets clear, i.e.

$$\begin{aligned} N c_1^* &= f(L_1^*) \\ N c_2^* &= f(L_2^*) \\ N k l_1^* &= L_1^* \\ N(k + g(i^*))l_2^* &= L_2^*. \end{aligned}$$

(ii) Is it possible for the price of consumption in the first period to be 1?

Answer: Yes. If $P^* = (p_1^*, p_2^*, w_1^*, w_2^*)$ is an equilibrium price vector, then so is P^*/p_1^*.

(iii) Write down a value function for the start of the second year. (Hint: the state variable includes human capital, savings, and the prices in the second year.)

Answer.

$$V(k_2, m_2; p_2, w_2) = \max_{c_2, l_2} u(c_2, l_2)$$
$$\text{s.t. } p_2 c_2 = w_2 k_2 l_2 + m_2.$$

(iv) Derive the marginal value of (a) human capital and (b) savings.

Answer. (a) Substituting the budget constraint into the objective gives

$$V(k_2, m_2; p_2, w_2) = u\left((w_2 k l_2(k_2, m_2; p_2, w_2) + m)/p_2, l_2(k_2, m_2; p_2, w_2)\right).$$

By the envelope theorem,

$$\begin{aligned}\frac{\partial V(k_2, m_2; p_2, w_2)}{\partial k} &= \left[\frac{\partial}{\partial k} u\left((w_2 k l_2 + m_2)/p_2, l_2\right)\right]_{l_2 = l_2(k_2, m_2; p_2, w_2)} \\ &= \left[u_c\left((w_2 k l_2 + m_2)/p_2, l_2\right) \frac{w_2 l_2}{p_2}\right]_{l_2 = l_2(k_2, m_2; p_2, w_2)} \\ &= u_c\left(c_2(k_2, m_2; p_2, w_2), l_2(k_2, m_2; p_2, w_2)\right) \frac{w_2 l_2(k_2, m_2; p_2, w_2)}{p_2}.\end{aligned}$$

(b) A similar procedure gives

$$\frac{\partial V(k_2, m_2; p_2, w_2)}{\partial m} = u_c\left(c_2(k_2, m_2; p_2, w_2), l_2(k_2, m_2; p_2, w_2)\right) \frac{1}{p_2}.$$

(v) The government thinks that it's wasteful for everybody to become educated. It proposes a tax on labour earnings in the second year to encourage more labour to be supplied in the first year. Could such a policy be Pareto-improving?

Answer. No. By the first-welfare theorem, the equilibrium (without any taxes) is efficient, so no Pareto-improving allocations are feasible.

(vi) * Informally discuss whether there are any asymmetric equilibria (e.g. in which some people choose to become well-educated, but others do not.)

Answer. Typically, the household's optimisation problem has a unique solution (because the objective is concave and the feasible choices lie in a convex set). When this is the case, all households have the same problem, and hence the same solution. In this model (as formulated in these sample solutions), the human capital multiplies hours worked in a non-convex way, so households might be indifferent between several choices. This could lead to multiple equilibria.

Question 12. (Micro 1 class exam in December 2014) A factory produces appliances using labour and waste disposal services. Households supply labour and waste disposal. Households are endowed with small or large gardens, where they can dispose of waste. Assume that households do not suffer from storing waste in their gardens, and that gardens are not traded (or at least, not directly).

(i) Write down a competitive model of the labour, appliance, and waste disposal markets.

Comment: A common mistake is to (implicitly) assume that households with big and small gardens made the same choices. You can't just write c for consumption, because people with bigger gardens will consume more. There are several alternatives. You could write c_h for household h's consumption, or you could write c_B for the big garden's consumption. (Or you could write the garden endowment as a parameter to the optimisation problem, and write down a policy function...) The most important thing is that the market clearing conditions (for all markets) accommodate people with different garden sizes making different choices.

It is also possible to formulate the consumer's problem so that the household can consume gardens in addition to selling them, e.g. by playing football. But they would not derive any utility from football, as per the question.

Answer: Consumer's problem. Notation: $h \in \{1, \ldots, N\}$ household address, a_h appliance choice, p price of appliances, l_h labour, w wages, g_h garden capacity, r price of disposal services, $u(a_h, l_h)$ utility, π firm profit (see below)

$$\max_{a_h, l_h} u(a_h, l_h) \tag{12}$$

$$\text{s.t. } pa_h = wl_h + rg_h + \pi/N. \tag{13}$$

Firm's problem. Notation: L labour demand, T waste supply, $A = f(L, T)$ appliance supply.

$$\pi(p, w, r) = \max_{L,T} pf(L, T) - wL - rT. \tag{14}$$

Market clearing conditions.

$$\sum_h a_h = A \tag{15}$$

$$\sum_h l_h = L \tag{16}$$

$$\sum_h g_h = T. \tag{17}$$

Equilibrium. A price vector (p^*, w^*, r^*) and an allocation

$$(\{a_h^*\}, \{l_h^*\}, A^*, L^*, T^*)$$

forms an equilibrium if the allocation satisfies the market clearing conditions, and the households' and firm's respective allocations solve their respective problems, given the price vector.

(ii) Show that in every equilibrium, all households' gardens are filled to capacity with waste.

Answer: If the price of waste disposal is greater than zero (i.e. $r > 0$), then there is a benefit, but no cost of filling the garden to capacity.

Alternative answer: If you formulate the household problem with a (useless) consumption choice of garden usage s_h, then an interior solution would satisfy the first-order condition for s_h,

$$0 = \lambda r,$$

where λ is the Lagrange multiplier on the budget constraint. Since $\lambda > 0$ and $r > 0$ in every equilibrium, this is a contradiction. So the assumption that s_h is an interior solution is false.

(iii) Show that if leisure is a normal good, then households with bigger gardens work less.

Answer: Households with bigger gardens have more wealth, and therefore consume more leisure (since leisure is a normal good). Which is another way of saying that they work less.

(iv) Show that if the price of waste disposal increases, then firms will generate less waste.

Answer: First, notice that π is the upper envelope of a set of straight lines, one for each choice (L, T). Therefore, π is convex. By the envelope theorem

$$\frac{\partial}{\partial r}\pi(p, w, r) = -T(p, w, r), \tag{18}$$

where $T(p, w, r)$ is the demand for waste disposal when prices are (p, w, r). Since π is convex, the left side is an increasing function in r. Therefore, the right side is also increasing in r, hence $T(p, w, r)$ is decreasing in r.

(v) Suppose the government wants to decrease the amount of waste stored in gardens. Is there a lump-sum tax scheme that would work?

Answer: No, by part (ii), no matter what the endowments are, all households will fill their gardens to capacity with waste. Therefore, there is no lump-sum tax regime that would work.

(vi) * Under what conditions would the households have a unique optimal labour, appliance and waste storage choice?

Answer: If all prices are non-zero, and the household's utility function is strictly quasi-concave (or strictly concave), then the household would have only one optimal choice.

(vii) * Prove that if all prices are greater than zero, and that households can work at most 24 hours per day, then the budget set (i.e. the set of affordable feasible choices) is compact.

Answer: We require $l \in [0, 24]$, so let $F = \mathbb{R}_+ \times [0, 24]$ be the set of feasible choices for the household (before considering the budget constraint).

Let $U_h(a, l) = wl + \pi/N + rg_h - pa$ be the amount of money that is unspent when household h chooses (a, l). This function is continuous. The set of affordable allocations is $A = (U_h)^{-1}(\mathbb{R}_+)$. Since $\mathbb{R}_+$ is closed and U_h is continuous, A is closed. The budget set $B = A \cap F$ is the intersection of two closed sets, and is therefore closed.

For any $(a, l) \in B$, we know $l \leq 24$, so $a \leq 24w + \pi/N + rg_h$. Therefore B is bounded, i.e. contained in some ball.

Since B is closed and bounded, the Bolzano-Weierstrass theorem implies that it is compact.

Question 13. (Micro 1 class exam in December 2014) As the earth's population grows, an important question is how future inhabitants will be able to feed themselves, and whether this will lead to inter-generational inequality. Suppose there are two generations (X and Y) of equal size. Generation X lives for two time periods, but Generation Y only lives in the second time period. This means that the population is higher in the second period.

Farms produce food using land and labour. Only Generation X is endowed with land, which it can supply to the market. Generation X households hold all of the shares in the farms. Both generations can supply labour and consume food. Households do not benefit from occupying land (but can gain wealth from renting out the land). Generation X has stationary time-separable preferences, and its per-period utility function is the same as Generation Y's.

(i) Write down a competitive general equilibrium model of this economy.

Comment: Firms are active in two time periods $t \in \{1, 2\}$. A common mistake is to write something like

$$\pi(p_t, w_t) = \max_{x_t} p_t f_t(x_t) - w_t \cdot x_t. \tag{19}$$

This is ambiguous, and both possible interpretations are wrong! One interpretation is that $\pi(p_t, w_t)$ is shorthand for $\pi(p_1, p_2, w_1, w_2)$. (A less ambiguous shorthand is $\pi(\{p_t, w_t\}_{t\in\{1,2\}})$ or just $\pi(p, w)$.) This interpretation makes no sense, because the objective does not explain how profits are combined from both periods. One way to fix this problem is to instead write

$$\pi(p, w) = \max_{x} \sum_{t\in\{1,2\}} \left[p_t f_t(x_t) - w_t \cdot x_t\right]. \tag{20}$$

Another interpretation is that there are two firms, one operating in each period. But if this is the case, they should have different profit functions, and in the households' budget constraints, you should be including the dividends of both firms. For example, you might write that the profit function of the firm operating in period t is

$$\pi^t(p_t, w_t) = \max_{x_t} p_t f_t(x_t) - w_t \cdot x_t. \tag{21}$$

Answer: Generation X's problem. Notation: c_t^X food consumption in period $t \in \{1, 2\}$, p_t food price, w_t wage, h_t^X labour supply, r_t land rent, l^X land endowment, $u(c, h)$ per-period utility function, β discount rate, π farm profit (see below), N^X Generation X population.

$$\max_{c_t^X, h_t^X} u(c_1^X, h_1^X) + \beta u(c_2^X, h_2^X) \tag{22}$$

$$\text{s.t. } p_1 c_1^X + p_2 c_2^X = w_1 h_1^X + w_2 h_2^X + (r_1 + r_2) l^X + \pi / N^X \tag{23}$$

Generation Y's problem. Notation: c^Y food consumption, h^Y labour supply, N^Y Generation Y population.

$$\max_{c^Y, h^Y} u(c^Y, h^Y) \tag{24}$$

$$\text{s.t. } p_2 c^Y = w_2 h^Y. \tag{25}$$

Farm's problem. Notation: L_t land demand, H_t labour demand, $C_t = f(L_t, H_t)$ food output in period t.

$$\pi(p_1, p_2, w_1, w_2, r_1, r_2) \tag{26}$$

$$= \max_{L_1, L_2, H_1, H_2} p_1 f(L_1, H_1) + p_2 f(L_2, H_2) - w_1 H_1 - w_2 H_2 - r_1 L_1 - r_2 L_2. \tag{27}$$

Market clearing conditions.

$$C_1 = N^X c_1^X \tag{28}$$

$$C_2 = N^X c_2^X + N^Y c^Y \tag{29}$$

$$L_1 = N^X l^X \tag{30}$$

$$L_2 = N^X l^X \tag{31}$$

$$H_1 = N^X h_1^X \tag{32}$$

$$H_2 = N^X h_2^X + N^Y h^Y. \tag{33}$$

Equilibrium. A price vector $(p_1, p_2, w_1, w_2, r_1, r_2)$ and an allocation

$$(\{(c_t^X, h_t^X)\}_t, c^Y, h^Y, \{(C_t, L_t, H_t)\}_t)$$

is an equilibrium if the households' and firms' allocations are optimal choices given the prices, and the markets clear.

(ii) Suppose that if the prices in all markets (labour, land, and food) do not increase over time, that there is excess demand of labour, land, and food in the second period. Does this imply that there is excess supply in all markets in the first period?

Answer: No. From Walras' law, we know that at least one market in the first period has excess supply, but it may not be all of them.

(iii) For this part, focus attention on equilibria in which food output is higher in the second period. Show that in every such equilibrium, real wages (i.e. wages divided by food prices) are lower in the second period.

Answer: In every equilibrium, $L_2 = L_1$ (from the market clearing conditions). Since food output is higher in the second period, this implies $H_2 > H_1$. From the firm's first-order conditions, we can deduce

$$f_H(L_1, H_1) = \frac{w_1}{p_1} \tag{34}$$

$$f_H(L_2, H_2) = \frac{w_2}{p_2}. \tag{35}$$

If f has decreasing marginal productivity, then $H_2 > H_1$ implies

$$f_H(L_1, H_1) > f_H(L_2, H_2). \tag{36}$$

We conclude then that real wages are higher in the first period, i.e.

$$\frac{w_1}{p_1} > \frac{w_2}{p_2}. \tag{37}$$

(iv) Write down Generation X's value of holding money in the second period. (Hint: this should be a function of money and second period food prices and wages.)

Answer: Generation X's indirect utility function is

$$v(m; p_2, w_2) = \max_{c_2^X, h_2^X} u(c_2^X, h_2^X) \tag{38}$$

$$\text{s.t. } p_2 c_2^X = m + w_2 h_2^X. \tag{39}$$

(v) Reformulate Generation X's problem by using the value function from (iv) twice, i.e. the household should choose how to allocate money between the two periods. How the money is spent in each period should be buried inside the value function.

Answer: A reformulation of the Generation X problem:

$$\max_{m_1, m_2} v(m_1; p_1, w_1) + \beta v(m_2; p_2, w_2) \tag{40}$$

$$\text{s.t. } m_1 + m_2 = \pi/N^X + (r_1 + r_2) l^X. \tag{41}$$

(vi) Generation Y protestors would like to eat more and work less, so they propose confiscating land from Generation X at the start of period 2, and giving it to Generation Y. Can such a policy make Generation Y better off? Would the proposal lead Generation Y to eat more and work less?

Answer: Confiscating land is equivalent to lump-sum taxation of the value of that land (at market prices). By the second welfare theorem, any efficient allocation can be implemented by doing this, and some efficient allocations would make Generation Y better off.

However, it's not clear if there is any efficient allocation in which Generation Y both works less and consumes more. (That depends on preferences.)

(vii) * The proof of existence of equilibrium relies on applying Brouwer's fixed point theorem, which requires a set to be convex (among other things). Economically speaking, which set is convex? Is this assumption usually met?

Answer: Brouwer's fixed point theorem is about a function $f : X \to X$, and it requires the set X to be convex. Economically speaking, X is the set of possible prices. The requirement that X be convex is very easy to satisfy. In the existence proof, we normalise prices to sum to 1, so the set of possible prices is a straight line (or hyperplane), which is convex.

(viii) * Holding prices fixed, consider a sequence of optimal labour supply and consumption choices, where the expenditure decreases to 1. Does this sequence have a convergent subsequence (using the Euclidean metric)?

Answer: Let e_n denote the expenditure for the n^{th} choice. Since e_n is decreasing, all choices are contained in the budget set corresponding to expenditure e_1. Since this budget set is compact, every sequence inside of it has a convergent subsequence.

Question 14. (Micro 1 degree exam in May 2015) Suppose there are two occupations, nursing and cleaning, and that individuals must select only one occupation to work in each year. Cleaning is easy to learn, but nurses with one year of experience become more productive. There are two years in the economy. Hospitals hire nurses and cleaners to provide medical services, and share their profits equally among the population. Individuals consume medical services.

(i) Write down a competitive model of the nursing and cleaning markets across the two years. (Hint: there are no symmetric equilibria, so you will need to accommodate identical households taking different decisions.)

Comment: This question is a little tricky to formulate well:

- One common mistake is to consider the experience a discrete choice, rather than depending on how hard the nurses work. This is partly my fault – it isn't until part (iv) that this becomes clear.
- The most common mistake is to write down the worker's utility functions conditional on occupation choice, but without studying the worker's decision about which occupation to choose. Despite this, students typically answer part (v) well (which was about workers being indifferent between nursing and cleaning)

Answer: Individuals. There are two fields, $o \in \{C, N\}$, cleaning and nursing. Individual $i \in I$ chooses how many hours to work in cleaning (h^i_{tC}) at wage w_{tC} and nursing (h^i_{tN}) at wage w_{tN}, consumption of medical m^i_t services at prices p_t. The experience-adjusted productivity of nursing in the second period is $x(h^i_{1N})$, where $x(0) = 1$. The individual has a discount factor β, and utility $u(m^i_t, 1 - h^i_{tC} - h^i_{tN})$ in each period. Hospital profits (defined below) are Π. Individual i's problem is:

$$\max_{\{m^i_t\}_t, \{h^i_{to}\}} \sum_{t=1}^{2} \beta^t u(m^i_t, 1 - h^i_{tC} - h^i_{tN})$$

$$\text{s.t. } p_1 m^i_1 + p_2 m^i_2 = w_{1C} h^i_{1C} + w_{1N} h^i_{1N} + w_{2C} h^i_{2C} + w_{2N} x(h^i_{1N}) h^i_{2N} + \frac{\Pi}{|I|},$$

and either $h^i_{tN} = 0$ or $h^i_{tC} = 0$.

The hospital. The hospital hires H_{tC} cleaner hours and H_{tN} productivity-adjusted nursing hours in time t, and produces $f(H_{tC}, H_{tN})$ units of medical services. Their profits are

$$\Pi(p_t, w_{1C}, w_{2C}, w_{1N}, w_{2N}) = \max_{H_{to}} \sum_t p_t f(H_{tC}, H_{tN}) - \sum_{t,o} w_{to} H_{to}. \tag{42}$$

Equilibrium. An allocation of resources $(\{m_t^{i*}, h_{tN}^{i*}, h_{tC}^{i*}\}, \{H_{to}^*\})$ and prices $(\{p_t^*\}, \{w_{to}^*\})$ constitute an equilibrium if each household and hospital finds this allocation optimal (see above), and the six markets clear, i.e.

$$\sum_i m_1^{i*} = f(H_{1C}^*, H_{1N}^*), \tag{43}$$

$$\sum_i m_2^{i*} = f(H_{2C}^*, H_{2N}^*), \text{ and} \tag{44}$$

$$\sum_i h_{to}^{i*} = H_{to}^* \text{ for } to \in \{1C, 1N, 2C, 2N\}. \tag{45}$$

(ii) Write down a formula for the value of savings and nursing experience in the second year.

Answer: Let s be savings, and x be nursing experience like before. Individual i's value function is

$$V_i(s, x) = \max_{m_2^i, h_{2C}^i, h_{2N}^i} u(m_2^i, 1 - h_{2C}^i - h_{2N}^i) \tag{46}$$

$$\text{s.t. } p_2 m_2^i = w_{2C} h_{2C}^i + w_{2N} x h_{2N}^i + s, \tag{47}$$

$$\text{and either } h_{2N}^i = 0 \text{ or } h_{2C}^i = 0. \tag{48}$$

(iii) Reformulate the year-one households' problem using the value function from the previous part.

Answer:

$$\max_{m_1^i, \{h_{1o}^i\}, s} u(m_1^i, 1 - h_{1C}^i - h_{1N}^i) + \beta V(s, x(h_{1N}^i)) \tag{49}$$

$$\text{s.t. } p_1 m_1^i + s = w_{1C} h_{1C}^i + w_{1N} h_{1N}^i + \frac{\Pi}{|I|}, \tag{50}$$

$$\text{and either } h_{tN}^i = 0 \text{ or } h_{tC}^i = 0. \tag{51}$$

(iv) What is the marginal value of nursing experience if the individual finds it optimal to do cleaning in the second year?

Answer: Zero. By the envelope theorem,

$$\frac{\partial V_i(s, x)}{\partial x} = \lambda w_{2N} h_{2N}^i, \tag{52}$$

where λ is the Lagrange multiplier for the budget constraint. If $h_{2N}^i = 0$, then the right side simplifies to 0.

(v) Argue informally that nurses have lower wages than cleaners in the first year.

Answer: Since some individuals choose each profession, all individuals are indifferent between being a cleaner and a nurse. Since nurses have a benefit (in the form of experience) in addition to wages, their wages must be lower in the first year.

(vi) Are competitive equilibria Pareto efficient in this economy? (Hint: list all the differences from pure-exchange economies where we proved the first-welfare theorem, and informally discuss whether these are important.)

Answer: Yes. The major differences are:

(a) **Production.** But home-production is equivalent.

(b) **Experience.** This is just another form of production.

(c) **Specialisation.** Individuals can only work in one occupation at a time. But this does not affect any part of the proof of the first welfare theorem. (The budget constraints can still be summed. Thus, we can show that an Pareto-improving allocation is worth more at market prices, and is therefore infeasible.)

(vii) * Is the excess demand function continuous?

Answer: No. At equilibrium prices, all households are indifferent between the two occupations. If the wage of cleaners increases slightly, then all households strictly prefer to specialise in cleaning, so there is a downwards jump in the excess demand of cleaners.

(viii) ** Is the household's feasiable choice set compact, assuming all prices are strictly greater than zero?

Answer: Yes. It is closed because it is the intersection of these two closed sets:

- Affordable allocations (because the budget constraint is continuous).
- The set of allocations involving at most one occupation.

It is bounded, because the number of working hours is limited, so the household's wealth is limited.

Question 15. (Micro 1 degree exam in May 2015) Suppose there are two schools that hire workers to teach. One school is twice as productive as the other – i.e. for the same amount of input, it produces double the output. Households supply labour and consume education.

(i) Write down a competitive model of this economy.

Comment. The most common mistake is getting confused about how many markets there are. The most straightforward approach is to assume there is a single labour market and a single education market. An alternative approach is to assume that these markets are separate, but that households value both types of education and labour/leisure equally. The households' first-order conditions would then imply that wages are equal in both markets, and education prices are equal in both markets.

Answer: Households. Hours h, wages w, education e, price of education p, utility $u(e, 1-h)$, school profits π_g and π_b (see below), n households. Household's problem is:

$$\max_{e,h} u(e, 1-h)$$
$$\text{s.t. } pe = wh + \frac{\pi_g + \pi_b}{n}.$$

Schools. School $s \in \{g, b\}$ has productivity factor $A_g = 2$ or $A_b = 1$, producing $A_s f(H)$ units of education from H hours of labour. The profit function of school s is

$$\pi_s(p, w) = \max_{H_s} pA_s f(H_s) - wH_s.$$

Equilibrium. $(h^*, e^*, H_g^*, H_b^*, p^*, w^*)$ is an equilibrium if these choices are optimal for each decision maker (as defined above), and markets clear, i.e.

$$nh^* = H_g^* + H_b^*$$
$$ne^* = A_g f(H_g^*) + A_b f(H_b^*).$$

(ii) Suppose at prevailing prices, there is excess supply of teachers. What does this imply about the supply of education?

Answer. By Walras' law, if there is excess supply in one market (of labour), then there is excess demand in another market. Since education is the only other market, we conclude there is excess demand for education.

(iii) Prove that the "good" (more productive) school hires more teachers than the "bad" school.

Answer. The school first-order condition is

$$pA_s f'(H_s) = w,$$

which can be rearranged to

$$f'(H_s) = \frac{w}{A_s p}.$$

Since $A_g > A_b$, the right side is smaller for the good school than the bad school. By decreasing marginal productivity, we conclude that $H_g > H_b$ in every equilibrium.

(iv) Prove that if wages increase, then schools provide less education.

Answer. By the envelope theorem,

$$\frac{\partial \pi_s(p, w)}{\partial w} = -H_s(p, w).$$

Now, π_s is the upper envelope of linear functions, so it is convex. Therefore the left side of the equation is increasing in w. It follows that $H_s(p, w)$ is decreasing in w. Total output

$$A_s f(H_s(p, w))$$

is therefore decreasing in w.

(v) Suppose that the government imposes lump-sum taxes on half of the population, and transfers these to the other half equally. Moreover suppose that education and leisure are normal goods, and that this policy causes real wages to increase. What happens to each household's education choices? Hint: the Slutsky equation is:

$$\underbrace{\frac{\partial x_i(p, m)}{\partial p_j}}_{\text{net effect}} = \underbrace{\left[\frac{\partial h_i(p, u)}{\partial p_j}\right]_{u=v(p,m)}}_{\text{substitution effect}} + \underbrace{\underbrace{-x_j(p, m)}_{\text{wealth lost}} \frac{\partial x_i(p, m)}{\partial m}}_{\text{income effect}}. \tag{53}$$

Comment. Most students struggled with this question, and overlooked that the previous part (iv) is a key ingredient. The Slutsky equation tells us about how individuals react, but the firm side of the market is also important for determining equilibrium outcomes.

Answer. By the previous part, schools supply less education, and demand less labour when real wages increase. Therefore, the total demand for education decreases.

Since real wages increased, the price of education (relative to wages) decreased. Therefore, the subsidised households have two changes to their budget constraint: the lump-sum transfer, and a price decrease of education. The first change increases wealth; this is a pure income effect which leads these households to demand more education. The second change is a price decrease in education; since education is a normal good (and hence not a Giffen good), this change leads households to consume (weakly) more education. The net effect of these changes is: the subsidised households demand more education.

Since the total demand for education decreases, the taxed households demand less education.

(vi) * In class, to prove the existence of an equilibrium, we constructed a continuous function and proved that it has a fixed point. Since we only need to consider one price in this economy (why?), this function effectively maps from $\mathbb{R}$ to $\mathbb{R}$. Describe mathematically, and sketch (i.e. draw) this function.

Answer. Since prices are relative, we can always normalise prices to sum to one. Therefore, we only need to think about one price – e.g. wages, w, since the other price is just $p = 1 - w$. By Walras' law, a wage of w forms an equilibrium if and only if the labour market clears at wage w.

Let $z_e(w)$ and $z_h(w)$ be the excess demand for education and labour, respectively. Let $Z_e(w) = \min\{z_e(w), 1\}$ and $Z_h(w) = \min\{z_h(w), 1\}$ be the truncated excess demand functions. (These are relevant when $w = 0$, which we must accommodate.)

Let $a_h(w) = \max\{0, Z_h(w)\}$ and $a_e(w) = \max\{0, Z_e(w)\}$ be the price adjustments for wages and education, respectively.

Consider the function

$$
\begin{aligned}
f(w) &= \frac{w + a_h(w)}{w + a_h(w) + (1 - w) + a_e(w)} \\
&= \frac{w + a_h(w)}{1 + a_h(w) + a_e(w)}.
\end{aligned}
$$

This function $f : [0, 1] \to [0, 1]$ is continuous. Moreover w^* is an equilibrium price if and only if w^* is a fixed point of f.

A sample graph is not included in these solutions.

(vii) ** Let (X, d) be any metric space. Prove that if $f, g : X \to \mathbb{R}$ are continuous, then $h(x) = \max\{f(x), g(x)\}$ is also continuous. Hint: you may assume a similar result holds for addition and subtraction.

Answer. (Note: this probably isn't the simplest possible proof...)

Recall that a function $\phi : X \to Y$ is continuous if for every closed set $U \subseteq Y$, the set $\phi^{-1}(U) \subseteq X$ is closed.

We can cut X into two sets:

$$X_f = \{x \in X : f(x) \geq g(x)\}$$
$$X_g = \{x \in X : g(x) \geq f(x)\}.$$

Note that X_f and X_g are closed in (X, d). (For example, $X_f = \Delta^{-1}(\mathbb{R}_+)$, where $\Delta(x) = f(x) - g(x)$.)

Since $X = X_f \cup X_g$, we can write

$$h^{-1}(U) = [h^{-1}(U) \cap X_f] \cup [h^{-1}(U) \cap X_g] \tag{54}$$
$$= [f^{-1}(U) \cap X_f] \cup [g^{-1}(U) \cap X_g]. \tag{55}$$

Since f is continuous, $f^{-1}(U)$ is closed. Moreover, the intersections of two closed sets is closed, so $[f^{-1}(U) \cap X_f]$ is closed. Similarly, the second set on the right side is closed. The union of two closed sets is closed. We conclude that $h^{-1}(U)$ is closed. Since this logic works for any closed set U, we have established that h is continuous.

Question 16. (Micro 1 class exam in December 2015) Consider a two-generation economy in which both generations consume fish in both time periods. However, the old generation can only work in the first period and the young can only work in the second period. A fishing firm hires workers in each period to catch fish, and a storage firm hires workers to freeze fish in the first time period, and to defrost fish in the second period. Defrosted and fresh fish are perfect substitutes.

(i) Write down a competitive model of the intergenerational fishing economy.

Comment. Many students struggle to formulate the storage firm's problem correctly. For example, many students did not require the storage firm to purchase fresh fish from the fishing firm.

Answer: Let $n = n^y + n^o$ be the total population, consisting of n^y young and n^o old.

Young households. Buys fish x_t^y in time t at price p_t, works h_2^y hours in period 2 at wages w_2, receives a share of the firms' profits $\Pi + \tilde{\Pi}$, gets utility $u^y(x_1^y, x_2^y, h_2^y)$ by:

$$\max_{x_1^y, x_2^y, h_2^y} u^y(x_1^y, x_2^y, h_2^y)$$
$$\text{s.t. } p_1 x_1^y + p_2 x_2^y = w_2 h_2^y + (\Pi + \tilde{\Pi})/n$$

Old households. Similarly,

$$\max_{x_1^o, x_2^o, h_1^o} u^o(x_1^o, x_2^o, h_1^o)$$
$$\text{s.t. } p_1 x_1^o + p_2 x_2^o = w_1 h_1^o + (\Pi + \tilde{\Pi})/n.$$

Fishing firm. Produces $f(H_t)$ fish from H_t hours of labour. Profit function:

$$\Pi(p_1, p_2, w_1, w_2) = \max_{H_1, H_2} p_1 f(H_1) + p_2 f(H_2) - w_1 H_1 - w_2 H_2.$$

Freezing firm. Produces $\tilde{f}(\tilde{X}_1, \tilde{H}_1, \tilde{H}_2)$ of unspoiled fish from $\tilde{H}_t$ hours of labour in period t and $\tilde{X}_1$ fresh fish. Profit function:

$$\tilde{\Pi}(p_1, p_2, w_1, w_2) = \max_{\tilde{X}_1, \tilde{H}_1, \tilde{H}_2} p_2 \tilde{f}(\tilde{X}_1, \tilde{H}_1, \tilde{H}_2) - p_1 \tilde{X}_1 - w_1 \tilde{H}_1 - w_2 \tilde{H}_2.$$

Equilibrium. An allocation $(x_1^y, x_2^y, h_2^y, x_1^o, x_2^o, h_1^y, H_1, H_2, \tilde{H}_1, \tilde{H}_2)$ and prices (p_1, p_2, w_1, w_2) form an equilibrium if these choices solve the households' and firms' problems above, and markets clear:

$$\begin{aligned} n^o h^o &= H_1 + \tilde{H}_1 \\ n^y h^y &= H_2 + \tilde{H}_2 \\ n^y x_1^y + n^o x_1^o + \tilde{X}_1 &= f(H_1) \\ n^y x_2^y + n^o x_2^o &= f(H_2) + \tilde{f}(\tilde{X}_1, \tilde{H}_1, \tilde{H}_2). \end{aligned}$$

(ii) Is it possible to normalise real wages in the first period to 1?

Answer. No. The real wage in the first period is w_1/p_1. If we multiply all prices by α, then the real wage is unchanged.

(iii) Show that if the price of fish in the second period increases, the storage firm sells more fish.

Answer. By the envelope theorem,

$$\begin{aligned} &\frac{\partial \tilde{\Pi}(p_1, p_2, w_1, w_2)}{\partial p_2} \\ &= \tilde{f}(\tilde{X}_1(p_1, p_2, w_1, w_2), \tilde{H}_1(p_1, p_2, w_1, w_2), \tilde{H}_2(p_1, p_2, w_1, w_2)) \\ &= \tilde{X}_2(p_1, p_2, w_1, w_2), \end{aligned}$$

where $\tilde{X}_2(p_1, p_2, w_1, w_2)$ is the optimal supply function.

Since $\tilde{\Pi}$ is the upper envelope of linear functions (one linear function for each production plan), it is convex. This means the left side of the equation above is increasing in p_2.

It follows that the right side of the equation – supply of fish in period two – is increasing in price p_2.

(iv) The government is worried about intergenerational inequality, i.e. that the young will receive lower real wages than the old. It proposes a lump-sum tax on the old and transfer to the young. Show if leisure is a normal good, then this causes at least some prices to change in the new equilibrium.

Comment. Most students are able to grasp the main intuition, but have difficulty writing a logical argument. The easiest way to formulate the answer is to do a proof by contradiction. "Suppose for the sake of argument, that no prices changed. Then, some impossible things would happen, so we can rule this out."

Answer. If the prices were the same, then the firms would choose the same production plans. This means the young would work the same amount, despite having more wealth (from transfers). This violates the assumption that leisure is a normal good.

(v) Suppose it is only possible to store whole fish. Are all equilibria Pareto efficient?

Comment. This question requires a discussion of the proof of the first welfare theorem. Specifically, does the proof rely on divisibility?

Answer. Yes, the proof of the first welfare theorem does not depend on divisibility. The main logic is that if there were a Pareto-dominating allocation, then it would have a higher market value, and therefore be infeasible.

(vi) * Suppose households can home-produce fish storage. Give an example of how this might lead household preferences to be time-inseparable.

Answer. The household might prefer not to buy fish tomorrow if it has fish stored from today. Specifically, consider the following four *market* choices of (x_1, h_1, x_2):

$$\begin{aligned} a &= (1, 1, 0), \\ b &= (1, 2, 1), \\ c &= (3, 1, 0), \\ d &= (3, 2, 1). \end{aligned}$$

The household might prefer $b \succ a$ and $c \succ d$, which violates time-separability.

(vii) ** Let (X, d_X) and (Y, d_Y) be metric spaces. Prove that if $f : X \to Y$ is continuous and X is compact in (X, d_X), then $f(X)$ is compact in (Y, d_Y).

Answer. We need to show that if $y_n \in f(X)$ is a sequence, then y_n has a convergent subsequence $y'_n \to_Y y'$.

Since each $y_n \in f(X)$, we know that there exists some $x_n \in X$ such that $y_n = f(x_n)$. Since X is compact, x_n has a convergent subsequence, $x'_n \to_X x'$. Let $y'_n = f(x'_n)$. Observe that y'_n is a subsequence of y_n.

Since f is continuous, $f(x'_n) \to_Y f(x')$, which means that $y'_n \to_Y f(x')$. We conclude that y'_n is a convergent subsequence of y_n, as required.

Question 17. (Micro 1 class exam in December 2015) Suppose that there are two time periods, and two seasons – summer and winter. There are about ten times as many people in the northern hemisphere than the southern hemisphere. This means that in both periods, an unequal fraction of people experience summer and winter. People prefer to work less and consume more in summer. A firm hires workers to produce a consumption good. It operates in both periods.

(i) Write down a competitive equilibrium model of seasons and hemispheres.

Comment. The main difficulty is capturing the differences between the Northern and Southern hemispheres. Many students confuse seasons and time – seasons are of course related to time, but they are not the same thing.

Answer: Let $n = n^N + n^S$ be the total population, consisting of n^N northern and n^S southern households. There are two periods $t \in \{1, 2\}$. In the first period, it is summer in the south, and winter in the north.

Households. A househould in location $\ell \in \{N, S\}$ has a discount rate of β^ℓ that depends on their location. We assume that $\beta^S < \beta^N$, which reflects the south's preference for higher consumption in the first period, etc.

Households consume $c_{\ell t}$ at price p_t, work $h_{\ell t}$ hours at wage w_t, which gives per-period utility $u(c_{\ell t}, h_{\ell t})$. Households receive dividends from firms' profits, Π. The household solves

$$\max_{\{c_{\ell t}, h_{\ell t}\}_{t=1}^{2}} u(c_{\ell 1}, h_{\ell 1}) + \beta^\ell u(c_{\ell 2}, h_{\ell 2})$$
$$\text{s.t. } p_1 c_{\ell 1} + p_2 c_{\ell 2} = w_1 h_{\ell 1} + w_2 h_{\ell 2} + \pi/n.$$

Firm. A single firm hire H_t hours of labour and produces $f(H_t)$ units of the consumption good in each period. Their profits are

$$\Pi(p_1, p_2, w_1, w_2) = \max_{H_1, H_2} p_1 f(H_1) + p_2 f(H_2) - w_1 H_1 - w_2 H_2.$$

Equilibrium. An allocation $(\{c_{\ell t}, h_{\ell t}\}_{t\in\{1,2\}, \ell\in\{N,S\}}, H_1, H_2)$ and prices (p_1, p_2, w_1, w_2) form an equilibrium if these choices solve the households' and firms' problems above, and markets clear:

$$\begin{aligned}
n^N h_{N1} + n^S h_{S1} &= H_1 \\
n^N h_{N2} + n^S h_{S2} &= H_2 \\
n^N c_{N1} + n^S c_{S1} &= f(H_1) \\
n^N c_{N2} + n^S c_{S2} &= f(H_2).
\end{aligned}$$

(ii) Suppose the market value of excess demand in all markets in the first time period is positive. Does this mean that there must be excess supply in a market in another time period?

Comment. This question is about Walras law, but a bit different from my usual questions. It's important to remember the big ideas behind all of the proofs – in this case "add up the households' budget constraints".

Answer. Yes. By Walras law, the market value of excess demand across the entire economy is 0. This means there must be some excess supply in other markets to cancel out the excess demand in the markets in the first period.

(iii) Using dynamic programming, reformulate the households' problems using net borrowing/lending as a state variable. That is, if this state variable is a positive number for period 1, then the household consumes more than its wages in period 1. The Bellman equation should bury the specifics about consumption or labour decisions in *both* periods.

Answer: Let $m_{\ell t}$ be the net resources devoted to period t by households in hemisphere ℓ. The households' indirect utility function can be reformulated as:

$$\begin{aligned} V_\ell(p_1, p_2, w_1, w_2) = \max_{m_{\ell 1}, m_{\ell 2}} \; & v(m_{\ell 1}; p_1, w_1) + \beta^\ell v(m_{\ell 2}; p_2, w_2) \\ \text{s.t. } & m_{\ell 1} + m_{\ell 2} = \pi/n, \end{aligned}$$

where

$$\begin{aligned} v(m, p, w) = \max_{c,h} \; & u(c, h) \\ \text{s.t. } & pc = wh + m. \end{aligned}$$

(iv) Show that households have a decreasing marginal value of net borrowing.

Answer: It suffices to show that $v(\cdot, p, w)$ is concave.

Suppose that u is concave. Suppose (c, h) is optimal for m, and (c', h') is optimal for m'. Then for any $\alpha \in (0, 1)$,

$$\begin{aligned} & v(\alpha m + (1-\alpha)m') \\ & \geq u(\alpha c + (1-\alpha)c', \alpha h + (1-\alpha)h') && \text{since this is affordable,} \\ & \geq \alpha u(c, h) + (1-\alpha)u(c', h') && \text{since } u \text{ is concave,} \\ & = \alpha v(m) + (1-\alpha)v(m'). \end{aligned}$$

(v) Show that households do more net borrowing (or less net lending) in summer than winter. *Hint: treat "how 'northern' a household is" as a state variable.*

Answer: Consider the value function

$$V(\beta, p_1, p_2, w_1, w_2) = \max_{m_1, m_2} v(m_1; p_1, w_1) + \beta v(m_2; p_2, w_2)$$
$$\text{s.t. } m_1 + m_2 = \pi/n.$$

This function is convex in β, because it is the upper envelope of a set of linear functions – one for each (m_1, m_2) choice. By the envelope theorem,

$$\frac{\partial V(\beta, p_1, p_2, w_1, w_2)}{\partial \beta} = v(m_2(\beta, p_1, p_2, w_1, w_2); p_2, w_2).$$

Since the left side is increasing in β, it follows that the right side is also increasing in β. Since v is increasing in resources m_2, it follows that the optimal policy $m_2(\beta, p_1, p_2, w_1, w_2)$ is increasing in β.

This means that southern households (low β) have low net borrowing m_2 in the second period (winter), while northern households (high β) have high net borrowing m_2 in the second period (summer). The reverse is true in period one, due to the budget constraint $m_1 + m_2 = \pi/n$.

Alternative Answer: The first-order condition for the optimal savings choices is:

$$v_1(m_{\ell 1}, p_1, w_1) - \beta^{\ell} v_1(\pi/n - m_{\ell 1}, p_2, w_2) = 0.$$

Let $m_1 = \phi(\beta)$ be the function that is implicitly defined by this equation, i.e. that gives the relationship between discounting and the optimal amount of resources to devote to the first period. By the implicit function theorem,

$$\phi'(\beta) = -\frac{-v_1(\pi/n - m_1, p_2, w_2)}{v_{11}(m_1, p_1, w_1) + \beta v_{11}(\pi/n - m_1, p_2, w_2)}$$

Now, $v_1 > 0$ and $v_{11} < 0$ (from the previous part), so we conclude that $\phi'(\beta) < 0$. Since we assumed that $\beta^S > \beta^N$, we conclude that $m_{S1} > m_{N1}$.

This means that southern households (low β) have high net borrowing m_1 in the first period (winter), while northern households (high β) have low net borrowing m_1 in the first period (summer). The reverse is true in period two, due to the budget constraint $m_1 + m_2 = \pi/n$.

(vi) The United Nations is worried that because of the population imbalance, the seasons create global inequality. They propose achieving equality by requiring everyone to work the same hours during summer and winter. Is it possible to design a lump-sum tax scheme that implements such an allocation? *Hint: assume that leisure is a normal good.*

Comment: Most students don't realise that the proposed allocation of resources is inefficient, so the second welfare theorem is inapplicable.

Answer: No, this is impossible. Any lump-sum tax scheme would not alter the conclusion from above that northern and southern households behave differently in terms of net borrowing/lending in the two time-periods. Since leisure is a normal good, they will still work different hours, as they have different effective income in each period and face the same prices as each other.

Since the second welfare theorem's conclusion does not hold, we conclude that its premise is false. That is, we conclude that the United Nations' target allocation is inefficient.

(vii) ** Prove that the boundary ∂A of any set A is closed.

Answer. We would like to show that if $x_n \in \partial A$ is a sequence and $x_n \to x^*$ then $x^* \in \partial A$.

Let $\varepsilon_n = d(x_n, x^*)$; note that $\varepsilon_n \to 0$. By taking an appropriate subsequence, we may assume without loss of generality that ε_n is decreasing.

Since $x_n \in \partial A$, there exists two sequences, $(a_n)_m \in A$ and $(b_n)_m \notin A$, both of which converge to x_n. There exists subsequences $(a'_n)_m$ and $(b'_n)_m$ such that $d((a'_n)_m, x_n) < \varepsilon_m$ and $d((b'_n)_m, x_n) < \varepsilon_m$.

Let $c_n = (a'_n)_n$ and $d_n = (b'_n)_n$. By the triangle inequality,

$$d(c_n, x^*) \leq d(c_n, x_n) + d(x_n, x^*).$$

I constructed these sequences so that $d(c_n, x_n) = d((a'_n)_n, x_n) < \varepsilon_n$, and $d(x_n, x^*) = \varepsilon_n$. I conclude that

$$d(c_n, x^*) < 2\varepsilon_n$$

and hence $c_n \to x^*$. Similarly, $d_n \to x^*$. Since $c_n \in \partial A$ and $d_n \notin \partial A$, it follows that $x^* \in \partial A$.

Alternative Answer. First, notice that $\partial A = \mathrm{cl}(A) \cap \mathrm{cl}(A^c)$, because $\mathrm{cl}(A)$ is the set of points that can be reached by taking the limit of a sequence inside A, and $\mathrm{cl}(A^c)$ is the set of points that can be reached by taking the limit of a sequence of points outside of A.

Now, the closure of any set is closed, so ∂A is the intersection of two closed sets. Therefore, ∂A is closed.

Question 18. (Micro 1 degree exam in May 2016) Scotland has two major cities, Glasgow and Edinburgh. Suppose that each city has an identical stock of buildings. Workers prefer to consume more buildings, and only benefit from housing located in the city that they choose to work in. There is an electronics factory in each city, that uses labour and buildings to produce electronics. The Glasgow factory is $z > 1$ times as productive as the Edinburgh factory (given the same inputs). To summarise, workers supply labour to factories, consume housing services in their own city, and consume electronics.

(i) Write down a competitive model of the Scottish housing and electronics economy.

Answer: Let n be the population of Scotland, and $\bar{B}$ be the building stock in each city $c \in C = \{\text{Edin}, \text{Glas}\}$.

Workers. Worker i consumes e_i electronics, $1 - h_i$ leisure, b_i buildings in city c_i. The price of electronics is p, the wage in city c is w_c, and the rent in city c is r_c. The worker's utility is $u(e_i, 1 - h_i, b_i)$. The worker owns an equal share of the building stock, $2B/n$, and in the two firms, whose profits are $\Pi = \Pi_{\text{Edin}} + \Pi_{\text{Glas}}$. The utility maximisation problem is:

$$\max_{c_i, e_i, h_i, b_i} u(e_i, 1 - h_i, b_i)$$
$$\text{s.t. } pe_i + r_{c_i} b_i = w_{c_i} h_i + (r_{\text{Edin}} + r_{\text{Glas}}) \frac{B}{n} + \frac{\Pi}{n}.$$

Firms. The factory in city c hires H_c workers, rents B_c buildings and produces $E_c = z_c f(H_c, B_c)$ items of electronics. The profit function is

$$\Pi_c(z_c, w_c, r_c) = \max_{H_c, B_c} p z_c f(H_c, B_c) - w_c H_c - r_c B_c.$$

Equilibrium. A price vector $(p, w_{\text{Edin}}, w_{\text{Glas}}, r_{\text{Edin}}, r_{\text{Glas}})$, a worker allocation $\{(c_i, e_i, h_i, b_i)\}_{i=1}^n$ and firm allocation $\{(H_c, B_c, E_c)\}_{c \in C}$ is an equilibrium if each worker's allocation solves the worker's problem, the firms' choices solve the firms' problems, and all markets clear, i.e.:

$$\sum_{i=1}^{n} e_i = E_{\text{Edin}} + E_{\text{Glas}}$$
$$\sum_{i=1}^{n} I(c_i = \text{Edin})h_i = H_{\text{Edin}}$$
$$\sum_{i=1}^{n} I(c_i = \text{Glas})h_i = H_{\text{Glas}}$$
$$\sum_{i=1}^{n} I(c_i = \text{Edin})b_i + B_{\text{Edin}} = \bar{B}$$
$$\sum_{i=1}^{n} I(c_i = \text{Glas})b_i + B_{\text{Glas}} = \bar{B}.$$

(ii) Suppose that there were excess demand for workers and housing in Glasgow, and that the electronics market cleared. Does this imply that there would be excess supply of workers and/or housing in Edinburgh?

Answer: Yes, there would either be excess supply of workers or housing in Edinburgh. By Walras' law, if there is excess demand in one market, there is excess supply in at least another market. By process of elimination, this must either be the labour or housing market in Edinburgh.

(iii) Prove that the Glasgow manufacturer's profit is increasing and convex in its productivity z.

Answer: The profit function in city c is

$$\Pi_c(z_c, w_c, r_c) = \max_{H_c, B_c} p z_c f(H_c, B_c) - w_c H_c - r_c B_c.$$

For each choice of (H_c, B_c), the objective is linear in z_c. Therefore, Π_c is the upper envelope of linear functions in z_c. We conclude that Π_c is convex in z_c.

(iv) Prove that if wages in Glasgow increase, then the Glasgow manufacturer demands fewer workers.

Answer: By the envelope theorem,

$$\frac{\partial}{\partial w_c}\Pi_c(z_c, w_c, r_c) = -H_c(z_c, w_c, r_c),$$

where $H_c(z_c, w_c, r_c)$ is firm c's labour demand curve.

By similar reasoning as in the previous part, Π_c is convex with respect to wages w_c (and building rents r_c). This means the left side of the above equation is increasing in w_c. We conclude that $H_c(z_c, w_c, r_c)$ is decreasing in w_c.

(v) Prove that if wages are higher in Glasgow, then rent is also higher in Glasgow.

Answer: Suppose for the sake of contradiction that wages are higher and rent is lower in Glasgow. If worker i chooses to live in Glasgow, his budget constraint is

$$(r_{\text{Edin}} + r_{\text{Glas}})\frac{B}{n} + w_{\text{Glas}}h_i + \frac{\Pi}{n} - pe_i - r_{\text{Glas}}b_i \geq 0$$

If worker i chooses to live in Edinburgh, his budget constraint is

$$(r_{\text{Edin}} + r_{\text{Glas}})\frac{B}{n} + w_{\text{Edin}}h_i + \frac{\Pi}{n}. - pe_i - r_{\text{Edin}}b_i \geq 0$$

The difference is

$$(r_{\text{Edin}} - r_{\text{Glas}})b_i + (w_{\text{Glas}} - w_{\text{Edin}})h_i.$$

By assumption, this difference is greater than zero. This implies that worker i is less constrained in Glasgow than Edinburgh, and hence prefers to move to Glasgow. But since some workers live in each city, all workers must be indifferent between Edinburgh and Glasgow. This is a contradiction.

(vi) Suppose there are several equilibria. Prove that every worker is indifferent between all equilibria.

Answer. In any equilibrium, all workers have the same utility as each other, since they have the same budget constraint and same utility function. Thus, if one worker is better off in a different equilibrium, then all workers are. But by the first welfare theorem, all equilibria are efficient. So no worker can be better off by switching to a different equilibrium.

(vii) * Prove that there is only one equilibrium allocation of resources.

Answer. By the previous part, in every equilibrium, all workers have the same utility. Therefore, by the first welfare theorem, the equilibrium allocation solves the social planner's problem,

$$\max_{E,\{H_c,B_c\},\{c_i,e_i,h_i,b_i\}_{i=1}^n} \sum_{i=1}^n u(e_i,h_i,b_i)$$

$$\text{s.t. } \sum_{i=1}^n e_i = z_{\text{Glas}} f(H_{\text{Glas}}, B_{\text{Glas}}) + z_{\text{Edin}} f(H_{\text{Edin}}, B_{\text{Edin}})$$

$$\sum_{i=1}^n I(c_i = \text{Glas}) b_i + B_{\text{Glas}} = \bar{B}$$

$$\sum_{i=1}^n I(c_i = \text{Edin}) b_i + B_{\text{Edin}} = \bar{B}$$

$$\sum_{i=1}^n I(c_i = \text{Glas}) h_i = H_{\text{Glas}}$$

$$\sum_{i=1}^n I(c_i = \text{Edin}) h_i = H_{\text{Edin}}.$$

The social planner's maximisation problem has a strictly concave objective, and a convex constraint set. Therefore, it has a unique solution. We conclude that there is only one equilibrium allocation.

(viii) ** Prove that if f and g are continuous, then $h(x) = f(g(x))$ is continuous.

Answer. There are many ways to prove this, using the various equivalent definitions of continuity. I will use the open set definition: a function $\phi : X \to Y$ is continuous if for every open subset $A \subset Y$, the set $\phi^{-1}(A)$ is an open subset of X.

Now, suppose $f : X \to Y$ and $g : Y \to Z$. This means $h : X \to Z$. Now, pick any open set A that is a subset of Z. Since g is continuous, $g^{-1}(A)$ is an open subset of Y. Since f is continuous, $f^{-1}(g^{-1}(A))$ is an open set of X. Now, $h^{-1}(z) = f^{-1}(g^{-1}(z))$, so we conclude that $h^{-1}(A)$ is an open set. Therefore, h is continuous.

Question 19. (Micro 1 degree exam in May 2016) According to Seixas, Robins, Attfield and Moulton (1992), coal miners have a 16% risk of developing the disease *black lung*. To keep things simple, suppose that all coal workers must retire early because of their health. Specifically suppose there are two time periods, and workers can choose to work in call centres or coal mines each period. After working in a coal mine, the worker is unable to work thereafter (in any job). However, sick retirees can still enjoy leisure as normal. A firm sells electricity, which it produces with coal miners and call centre workers. Workers supply either kind of labour and consume electricity and leisure.

(i) Write down a competitive model of the electricity markets and the two types of labour markets.

Answer.

Households. Household $h \in H$ chooses their job $j_{ht} \in J = \{m, c\}$ in time $t \in T = \{1, 2\}$, where m is mining and c is call centre, their labour supply l_{ht} in time t, and electricity consumption e_{ht} in time t, The prices are w_{jt} and p_t respectively. These choices give utility $\sum_{t \in T} \beta^t u(e_{ht}, 1 - l_{ht})$. The household's problem is

$$\begin{aligned} &\max_{\{j_{ht}, e_{ht}, l_{ht}\}_t} \sum_{t \in T} \beta^t u(e_{ht}, 1 - l_{ht}) \\ &\text{s.t. } \sum_{t \in T} p_t e_{ht} = \sum_{t \in T} w_{j_{ht}t} l_{ht} + \frac{\Pi}{|H|}, \\ &I(j_{h1} = m) l_{h2} = 0, \end{aligned}$$

where Π is the firm's profits (see below).

Firm. The firm chooses the number of miners M_t and call centre workers C_t, which enables it to supply $E_t = f(M_t, C_t)$ units of electricity. Its profit function is

$$\begin{aligned} &\Pi(w_{1m}, w_{1c}, w_{2m}, w_{2c}, p_1, p_2) \\ &= \max_{M_1, C_1, M_2, C_2} p_1 f(M_1, C_1) + p_2 f(M_2, C_2) - w_{1m} M_1 - w_{2m} M_2 - w_{1c} C_1 - w_{2c} C_2. \end{aligned}$$

Equilibrium. A price vector $(w_{1m}, w_{1c}, w_{2m}, w_{2c}, p_1, p_2)$, a worker allocation $\{j_{ht}, e_{ht}, l_{ht}\}_{t,h}$ and a firm allocation $(M_1, C_1, E_1, M_2, C_2, E_2)$ form an equilibrium if each worker's allocation solves the worker's problem, the firm's choices solve the firm's problems, and all markets clear, i.e.:

$$\sum_{h \in H_{m1}} l_{h1} = M_1$$
$$\sum_{h \in H_{m2}} l_{h2} = M_2$$
$$\sum_{h \in H_{c1}} l_{h1} = C_1$$
$$\sum_{h \in H_{c2}} l_{h2} = C_2$$
$$\sum_{h \in H} e_1 = E_1$$
$$\sum_{h \in H} e_2 = E_2.$$

where $H_{j't} = \{h \in H : j_{ht} = j'\}$ is the set of households who do job j' in period t.

(ii) Reformulate the worker's problem with a Bellman equation, using wealth and health as state variables.

Answer. Let x denote wealth and $y \in \{0, 1\}$ denote health, where $y = 1$ denotes good health. The last period value function is:

$$\begin{aligned} V(x, y) = &\max_{j,e,l} u(e, 1-l) \\ &\text{s.t. } p_2 e = w_{j2} l y + x. \end{aligned}$$

The household's problem can be written as

$$\begin{aligned} &\max_{j,e,l,x'} u(e, 1-l) + \beta V(x', I(j = c)) \\ &\text{s.t. } p_1 e + x' = w_{j1} l + \frac{\Pi}{|H|}. \end{aligned}$$

(iii) Prove that in the last period, both professions receive the same wage.

Answer. Looking at the last period value function, the only difference between the jobs is the wage w_{j2}. If the wage in one profession were higher, then all workers would work in that profession. But then the market for the other profession would not clear (the firm will always demand some workers for each job, e.g. if production is impossible without some of each).

(iv) Prove that the worker has diminishing marginal value of wealth in the last period.

Answer. Since the wages in the last period are equal, the choice j is immaterial, so that

$$V(x, y) = \max_{e,l} u(e, 1 - l) \quad \text{s.t. } p_2 e = w_{m2} l y + x.$$

Fix $y = y'$, and suppose that (e, l) are optimal at (x, y) and (e', l') are optimal at (x', y'). Then

$$\begin{aligned} &\alpha V(x, y) + (1 - \alpha)V(x', y') \\ &= \alpha u(e, 1 - l) + (1 - \alpha)u(e', 1 - l') \\ &\leq u(\alpha(e, 1 - l) + (1 - \alpha)(e', 1 - l')) \\ &\leq V(\alpha(x, y) + (1 - \alpha)(x', y')). \end{aligned}$$

Therefore, V is concave in x, so the household has a diminishing marginal value of savings, i.e. $\partial V/\partial x$ is decreasing in x.

(v) Prove that in the first period, coal miners receive higher wages than call centre workers.

Answer. Since unhealthy workers can't earn labour income in the second period, we know that $V(x, 1) > V(x, 0)$ for all x. Thus, mining imposes a cost of $V(x, 1) - V(x, 0)$ on the worker. For the worker to be indifferent between the two jobs, the mining wage w_{m1} must be higher than the call centre wage w_{c1}.

(vi) Suppose the government selects half of the population (e.g. those born in the first half of the year) for a reward, to be funded by lump-sum taxes on the other half of the population. Is this policy Pareto efficient?

Answer. Yes. The lump-sum transfers are equivalent to re-arranging the endowments. The first welfare theorem establishes that (regardless of the endowment) all equilibria are Pareto efficient.

(vii) ** Consider the metric space (X, d) where $X = [0, 2]$ and $d(x, y) = |x - y|$. Prove or disprove that $A = [0, 1)$ is an open set.

Answer. A is an open set.

Recall that A is open if for every point $a \in A$, there is an open neighbourhood $N_r(a) = \{b \in X : d(a, b) < r\}$ centred at a with a radius of $r > 0$ such that $N \subseteq A$.

For any point a, we can select $r = d(a,1) = 1 - a$. With this choice of r, we need to check that $N_r(a) \subseteq A$.

Suppose $b \in N_r(a)$. Then $b \in [0,2]$ and $d(a,b) < 1 - a$. This leads to two possibilities: $b \in [0,a]$ or $b \in (a,2]$. For the first possibility, since $[0,a] \subseteq A$, we conclude $b \in A$. For the second possibility, $d(a,b) = b - a < 1 - a$, so that $b < 1$ and hence $b \in A$.

Question 20. (Advanced Mathematical Economics mock exam)

Part A. Parts (i), (ii), (iii), and (iv) of Question 19.

Part B.

(i) Let $X = \{(x, y) \in \mathbb{R}^2 : x + y \leq 1\}$. What is the boundary of the set

$$A = \left\{(x, y) \in \mathbb{R}^2_+ : x + y \leq 1\right\}$$

inside the metric space (X, d_2)?

Answer. The boundary is

$$\partial A = \{(x, 0) : x \in [0, 1]\} \cup \{(0, y) : y \in [0, 1]\}$$

because

- each point z in this set is inside A (so the sequence $a_n = z \in A$ has $a_n \to z$), and
- a point $(x, 0)$ in this set has a sequence $b_n = (x, -1/n) \in X \backslash A$ where $b_n \to (x, 0)$, and similarly for $(0, y)$.

(ii) Consider the sequence of functions $f_n \in CB([0, 1])$ defined by $f_n(x) = x + x/n$. Is f_n a convergent sequence in $(CB[0, 1], d_\infty)$?

Answer. Yes, $f_n \to f^*$ where $f^*(x) = x$, because $d_\infty(f_n, f^*) = d_1(f_n(1), f^*(1)) = |1/n - 1| \to 0$.

(iii) Prove that $(l_\infty([0, 1]), d_\infty)$ is not a compact metric space. (Recall that $l_\infty([0, 1])$ is the set of bounded sequences $x_n \in [0, 1]$.) *Hint: you only need to find one counterexample.*

Answer. In a compact metric space, every sequence has a convergent subsequence. Now, consider the sequence

$$(x_n)_m = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

Notice that $d_\infty(x_n, x_{n'}) = 1$ for all $n \neq n'$. Therefore, x_n has no convergent subsequence. We conclude that $(l_\infty([0, 1]), d_\infty)$ is not compact.

(iv) Consider any metric space (X, d). Let $x_n, y_n, z_n \in X$ be sequences. Suppose $x_n \to x^*$ and $z_n \to x^*$. Prove that if $d(x_n, y_n) \leq d(x_n, z_n)$ for all n then $y_n \to x^*$.

Answer. Pick any $r > 0$. We would like to show that there exist some N such that

$$d(y_n, x^*) < r \text{ for all } n > N.$$

Since $x_n \to x^*$, there is some number N_x such that

$$d(x_n, x^*) < r/3 \text{ for all } n > N_x.$$

Since $z_n \to x^*$, there is an analogous number N_z.

Let $N = \max\{N_x, N_z\}$. Then,

$$\begin{aligned}
d(y_n, x^*) &\leq d(x_n, y_n) + d(x_n, x^*) && \text{(triangle inequality)} \\
&\leq d(x_n, z_n) + d(x_n, x^*) && \text{(assumption)} \\
&\leq d(x_n, x^*) + d(z_n, x^*) + d(x_n, x^*) && \text{(triangle inequality)} \\
&\leq r/3 + r/3 + r/3 \\
&= r
\end{aligned}$$

for all $n > N$. We conclude that $y_n \to x^*$.

(v) Write down a recursive Bellman equation for an infinite horizon cake-eating problem in which the size of the cake grows by $r = 0.01 \times 100\%$ every day. Prove that the Bellman operator a contraction on $(CB(\mathbb{R}), d_\infty)$. What is the degree of the contraction? (You do not need to prove that the Bellman operator is a self-map.)

Answer. An appropriate Bellman equation is:

$$\begin{aligned}
V(k) = \max_{x, k' \geq 0} &\; u(x) + \beta V(k') \\
\text{s.t. } &\; x + k' \geq k(1 + r).
\end{aligned}$$

Let $F(V)(k)$ be the right side of the above, i.e. the Bellman operator. We now show that F is a contraction. Consider any two value functions $V_1, V_2 \in CB(\mathbb{R}_+)$. Let $x_1(k)$ and $x_2(k)$ be the corresponding policy functions. Then:

$$\begin{aligned}
F(V_1)(k) &= u(x_1(k)) + \beta V_1([k - x_1(k)]/(1 + r)) \\
&= u(x_1(k)) + \beta V_2([k - x_1(k)]/(1 + r)) \\
&\quad - \beta V_2([k - x_1(k)]/(1 + r)) + \beta V_1([k - x_1(k)]/(1 + r)) \\
&\leq u(x_1(k)) + \beta V_2([k - x_1(k)]/(1 + r)) + \beta d_\infty(V_1, V_2) \\
&\leq [u(x_2(k)) + \beta V_2([k - x_2(k)]/(1 + r))] + \beta d_\infty(V_1, V_2) \\
&= F(V_2)(k) + \beta d_\infty(V_1, V_2)
\end{aligned}$$

This implies that:

$$F(V_1)(k) - F(V_2)(k) \leq \beta d_\infty(V_1, V_2) \text{ for all } k \geq 0.$$

Reversing the roles of V_1 and V_2 gives the inequality:

$$F(V_2)(k) - F(V_1)(k) \leq \beta d_\infty(V_1, V_2).$$

Together, these imply that

$$d_\infty(F(V_2), F(V_1)) \leq \beta d_\infty(V_1, V_2).$$

We conclude that F is a contraction of degree β.

(vi) Let (X, d) be a complete metric space, and $f : X \to X$ be a continuous function. Fix any $x_0 \in X$, and consider the sequence $x_{n+1} = f(x_n)$. Prove that if x_n is a Cauchy sequence then x_n converges to a fixed-point of f.

Answer. This is an important part of the proof of Banach's fixed point theorem.

Since (X, d) is complete, x_n is a convergent sequence. Let x^* be its limit. Since f is continuous, $f(x_n) \to f(x^*)$. Since $x_{n+1} = f(x_n)$, the sequence $f(x_n)$ is a subsequence of x_n, so $f(x_n) \to x^*$. Since $f(x_n)$ converges both to x^* and $f(x^*)$, we conclude that $x^* = f(x^*)$. Therefore, x_n converges to a fixed-point of f.

(vii) Let $X = \{f \in CB([0,1]) : f(x) = ax \text{ for some } a \in [0,1]\}$. Is (X, d_∞) a compact metric space?

Answer. Yes. Let $F(a) = (x \mapsto ax)$, where $F : \mathbb{R} \to CB([0,1])$. First, F is continuous: if $a_n \to a^*$ then $d_\infty(F(a_n), F(a^*)) = \sup_{x\in[0,1]} |a_n x - a^* x| = |a_n - a^*| \to 0$. Second, $[0,1]$ is a compact set in $(\mathbb{R}, d_2)$. In class, we proved that if F is continous and A is compact then $F(A)$ is compact. Therefore, $X = F([0,1])$ is compact.

(viii) Prove that the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at $x^* = 0$.

Answer. Since $\sin(1/x) \in [-1, 1]$, we know that $-x^2 \leq f(x) \leq x^2$ for all x. By the differentiable sandwich lemma, f is differentiable at $x^* = 0$.

Question 21. (Advanced Mathematical Economics December 2016)

Part A

CAF (Construcciones y Auxiliar de Ferrocarriles) produces trams and replacement parts for Edinburgh Trams using labour. Suppose that for each tram used in the first year of operation, 0.2 trams worth of parts must be bought for maintenance before the tram can be used in the second year. Edinburgh Trams produces public transport services from trams and labour to households. Households supply labour to the two companies and consume transport. All households have the same preferences, and shares in all firms are shared equally among all households.

(i) Write down a general equilibrium model of the labour, tram and transportation markets involving households, the factory, and the tram operator over a two-year period. (Hint: Pay careful attention to the depreciation of trams.)

Comment. The main difficulty students had was correctly including the depreciated stock of the first year's trams into the second year.

Answer. Household's problem. There are n households, each of which receives a portion $\pi_{CAF}/n + \pi_{ET}$ of the firms' profits. Households choose working hours h_t and journeys j_t in each time period $t \in \{1, 2\}$ at prices w_t and p_t respectively. This gives the household utility $u(h_1, j_1) + \beta u(h_2, j_2)$, where u is the flow utility function. The household's problem is

$$\max_{h_1,h_2,j_1,j_2} u(h_1, j_1) + \beta u(h_2, j_2)$$
$$\text{s.t. } p_1 j_1 + p_2 j_2 = w_1 h_1 + w_2 h_2 + \pi/n.$$

Edinburgh Tram's problem. Edinburgh Trams purchases Y_t units of trams at a cost of r_t in period t, so that the stock is $K_1 = Y_1$ in the first year, and $K_2 = (1 - \delta)K_1 + Y_2$ in the second year, where depreciation rate is $\delta = 0.2$. They hire H_t hours of labour and sell $J_t = f(K_t, H_t)$ journeys in year t. Edinburgh Trams' profits are

$$\begin{aligned}&\pi_{ET}(p_1, p_2; r_1, r_2, w_1, w_2)\\ &= \max_{Y_1,Y_2,H_1,H_2} p_1 f(Y_1, H_1) + p_2 f((1 - \delta)Y_1 + Y_2, H_2) - w_1 H_1 - w_2 H_2 - r_1 Y_1 - r_2 Y_2.\end{aligned}$$

CAF's problem. CAF purchases H'_t units of labour in period t to produce $Y'_t = g(H'_T)$ trams. Its profits are

$$\begin{aligned}&\pi_{CAF}(r_1, r_2; w_1, w_2)\\ &= \max_{H'_1,H'_2} r_1 g(H'_1) + r_2 g(H'_2) - w_1 H'_1 - w_2 H'_2.\end{aligned}$$

Equilibrium. A vector of prices $(r_1, r_2, p_1, p_2, w_1, w_2)$ and quantities

$$(h_1, h_2, j_1, j_2, J_1, J_2, Y_1, Y_2, H_1, H_2, Y_1', Y_2', H_1, H_2)$$

form an equilibrium if they solve the problems above, and supply equals demand in each market:

$$\begin{aligned} Y_1' &= Y_1 \\ Y_2' &= Y_2 \\ nh_1 &= H_1 + H_1' \\ nh_2 &= H_2 + H_2' \\ nj_1 &= J_1 \\ nj_2 &= J_2. \end{aligned}$$

(ii) Write down a Bellman equation for Edinburgh Trams' decision in the first year that buries the second year choices in a value function.

Comment. The key to this question is that the trams from last year is a state variable. Even if you messed up the first part, you could still try to find a way to include it here.

Answer. Let

$$\begin{aligned} &\pi_{ET2}(K_1; p_2; r_2, w_2) \\ &= \max_{Y_2, H_2} p_2 f((1-\delta)K_1 + Y_2, H_2) - w_2 H_2 - r_2 Y_2. \end{aligned}$$

Then the Edinburgh Trams' profit function can rewritten with a Bellman equation:

$$\begin{aligned} &\pi_{ET}(p_1, p_2; r_1, r_2, w_1, w_2) \\ &= \max_{Y_1, H_1} p_1 f(Y_1, H_1) - w_1 H_1 - r_1 Y_1 + \pi_{ET2}(Y_1; p_2, r_2, w_2). \end{aligned}$$

(iii) Show that Edinburgh trams' second year value function is convex in prices.

Answer. Specifically, we need to show that for each capital stock K_1, $\pi_{ET2}(K_1; \cdot; \cdot, \cdot)$ is a convex function. Now, π_{ET2} is the upper envelope of a set of convex functions. Specifically, each choice of (Y_2, H_2) has a corresponding linear (and hence convex) function

$$(p_2; w_2, r_2) \mapsto p_2 f((1-\delta)K_1 + Y_2, H_2) - w_2 H_2 - r_2 Y_2.$$

Since upper envelopes of convex functions are convex, we conclude that $\pi_{ET2}(K_1; \cdot; \cdot, \cdot)$ is a convex function.

(iv) Show that if the price of trams increases in the second year, then Edinburgh Trams buys fewer trams in the second year.

Answer. By the envelope theorem,

$$\frac{\partial \pi_{ET2}}{\partial r_2} = -Y_2(K_1; p_2; w_2, r_2).$$

We established above that π_{ET2} is convex in r_2, so the left side is increasing in r_2. It follows that the right side is increasing in r_2, i.e. that if the price of trams increases, then Edinburgh Trams buys fewer trams in the second year.

Part B

(i) Let (X, d) be a metric space and let $A \subseteq X$. Prove that the boundary of A is a closed set.

Comment. This is a hard question. However, it appeared in a previous exam (question 17.vii), so I was surprised that few students answered it correctly. Most students made fundamental mistakes when trying to answer this. For example, students might write that if $a \in \partial A$, then $a \in A$ (not true) or if $a_n \in A$ and $a_n \to a^*$ then $a^* \in \partial A$ (also not true).

Another common mistake was that student wrote that if $x \in \partial A$, then $\{x\}$ is a closed set (correct) and therefore the union of all of these sets is closed (incorrect). A union of a *finite* collection of closed sets is closed, but not an infinite set. For example, $\cup_n [0, (n-1)/n] = [0, 1)$ is not closed.

Answer. A short and clever answer is available in (17.vii). Here is a less creative solution.

Suppose $x_n \in \partial A$ is a convergent sequence with $x_n \to x^*$. We want to show that $x^* \in \partial A$. Specifically, we want to show that there are two sequences, namely:

- $a_n \in A$ with $a_n \to x^*$ and
- $b_n \in X \backslash A$ with $b_n \to x^*$.

Without loss of generality, assume that $d(x_n, x^*) < 1/n$. Fix any n. Since $x_n \in \partial A$, there is a sequence $\hat{a}_m \in A$ with $\hat{a}_m \to x_n$. Therefore, there exists some $a_n \in A$ such that $d(a_n, x_n) \leq 1/n$.

Now, by the triangle inequality, $d(a_n, x^*) \leq d(a_n, x_n) + d(x_n, x^*) = 1/n + 1/n \to 0$. Therefore, $a_n \to x^*$.

A similar argument establishes that there exists some $b_n \in X \backslash A$ such that $d(b_n, x_n) \leq 1/n$ hence $b_n \to x^*$.

(ii) Suppose (X, d) is a compact metric space. Prove that if $A \subseteq X$ is a closed set, then A is a compact set.

Comment. Most students answered this question well. However, many students made mistakes. For example, students wrote that since (X, d) is compact, the limit of $a_n \in A$ has to lie in X. This reflects two misunderstandings: (1) that the definition of "limit" only makes sense if the limit is inside the metric space's point set X, and (2) compactness only implies that a_n has a convergent subsequence; a_n itself need not be convergent.

Similarly, some students wrote that since X is compact, every subsequence of $a_n \in X$ is convergent (not true). Of course, compactness only requires that *at least one* subsequence of a_n is convergent, not *every* subsequence. If compactness required that every subsequence of a_n be convergent, then this would imply a_n itself be a convergent sequence, since a_n is a subsequence of itself.

Answer. This is an alternative to the proof given in the lecture notes.

Suppose $a_n \in A$ is a sequence. Since $a_n \in X$, it has a convergent subsequence $b_n \in A$ with $b_n \to b^*$. Moreover, since A is closed, $b^* \in A$. Thus, we have shown that a_n has a convergent subsequence whose limit lies in A. We conclude that A is compact.

(iii) Let (A, d) be a compact metric space. Consider an optimisation problem:

$$\max_{a \in A} u(a),$$

where $u : A \to \mathbb{R}$ is continuous. Prove that the set of optimal choices,

$$A^* = \{a \in A : u(a) \geq u(a') \text{ for all } a' \in A\}$$

is compact. *Hint: use the previous question.*

Comment. My impression here is that most people had a good idea about how to prove this, but were unable to express their idea because of difficulties with mathematical notation. There are two tricks to take notice of in my answer. First, I pick out one optimal choice, and give it a name – a^*. Giving things a name is very helpful, because it means you can refer back to it in a precise way later on. Similarly, I pick out the optimal utility level, and give that a name too – u^*. This is a general lesson: *give mathematical names to important things.*

Second, I connect the set of optimal choices A^* to the utility function u via the formula $A^* = u^{-1}(\{u^*\})$. This is made much simpler because of my first trick.

Answer. By the Weierstrass theorem, there is at least one optimal choice, a^*, which gives utility $u^* = u(a^*)$. The set of all optimal choices is $A^* = u^{-1}(\{u^*\})$. Since u is continuous and $\{u^*\}$ is a closed set, it follows that A^* is a closed set. The previous question established that closed subsets of compact metric spaces are compact. Since $A^* \subseteq X$ is closed and X is compact, we conclude that A^* is compact.

(iv) Prove that $(CB(\mathbb{R}), d_\infty)$ is *not* a compact metric space. *Hint: you only need one counterexample.*

Comment. Students gave many creative counterexamples to this question. Complicated answers are just as valid (i.e. compelling evidence) as simple ones, but I encourage students to think about the simplest possible answers.

Answer. The sequence of functions $f_n(x) = n$ has no convergent subseqeunce, because $d(f_n, f_m) \geq 1$ for $n \neq m$.

(v) Suppose that the stock of salmon in the North Sea naturally doubles every five years. Individuals enjoy eating salmon according to a discounted utility function. (a) Write down a recursive Bellman equation to represent the social planner's problem over an infinite time horizon. (b) Sketch a proof that the social value of the stock of salmon is a continuous function. (You do not need to prove that the Bellman operator is a contraction, or prove the principle of optimality.)

Comment. Many students did not understand what *recursive* means – it means that it's the same value function on both sides, i.e. V and V, not V_1 and V_2. Most students did not think of using Banach's fixed point theorem for part (b).

Answer. Let k be the stock of salmon, x be salmon consumption, $1 + r$ be the rate of natural growth, β the discount rate, $u(x)$ be the flow value of consuming Salmon, and $V(k)$ be the social value of salmon,

$$\begin{aligned} V(k) = & \max_{x,k'} u(x) + \beta V(k'(1+r)) \\ & \text{s.t. } x + k' = k. \end{aligned}$$

The corresponding Bellman operator,

$$\begin{aligned} F(V)(k) = & \max_{x,k'} u(x) + \beta V(k'(1+r)) \\ & \text{s.t. } x + k' = k \end{aligned}$$

is a contraction on the complete metric space, $(CB(\mathbb{R}_+), d_\infty)$. Therefore F has a unique fixed point, V, that is a continuous and bounded function. Since

V is a fixed point of the Bellman operator, it solves the Bellman equation. By the principle of optimality, $V(k)$ is the social value of a salmon stock of k. We conclude that the social value of salmon stocks is a continuous function.

(vi) Let $f : X \to X$ be a function on the metric space (X, d). Prove that if f has two fixed points, $x^* \neq x^{**}$, then f is not a contraction.

Answer. Suppose for the sake of contradiction that f were a contraction of degree $a < 1$. Then $d(f(x^*), f(x^{**})) \leq ad(x^*, x^{**}) < d(x^*, x^{**})$. Since x^* and x^{**} are fixed points of f, we have $d(f(x^*), f(x^{**})) = d(x^*, x^{**})$. These two conclusions are contradictory.

(vii) Let

$$u(x, y) = \frac{x + y}{1 + y^2 - \sqrt{y}},$$

where $(x, y) \in \mathbb{R}_+ \times [0, 1]$. Find a differentiable lower support function at $x = 2$ for

$$f(x) = \max_{y \in [0,1]} u(x, y).$$

Comment. One trick here is that it's not necessary to solve for the optimal choice at $x = 2$. It suffices to prove that there is an optimal choice, and then just pick one and give it a name (I called it y^*).

Answer. Let y^* be an optimal choice at $x^* = 2$. (There is an optimal choice, since the objective is continuous and the choice set is compact.)

Consider the function $L(x) = \frac{x+y^*}{1+(y^*)^2-\sqrt{y^*}}$. This function is linear, and therefore differentiable. Moreover, $L(2) = f(2)$ and $L(x) = u(x, y^*) \leq \max_y u(x, y) = f(x)$ for all $x \in \mathbb{R}_+$. Therefore, L is a differentiable lower support function for f.

(viii) Suppose that $f : \mathbb{R}_+^{N-1} \to \mathbb{R}$ is strictly concave. Prove that there is at most one solution to the profit maximisation problem,

$$\max_{x \in \mathbb{R}_+^{N-1}} pf(x) - w \cdot x$$

where $(p, w) \in \mathbb{R}_{++}^N$.

Comment. Students often had good intuition, but could not convert that into a general proof. I suggest writing down relevant definitions and/or theorems to get started in writing down a proof.

Answer. Suppose for the sake of contradiction that $x^* \neq x^{**}$ are both solutions, so that

$$\pi^* = pf(x^*) - w \cdot x^* = pf(x^{**}) - w \cdot x^{**}.$$

Consider $\hat{x} = \frac{1}{2}(x^* + x^{**})$. Then,

$$\begin{aligned}
& pf(\hat{x}) - w \cdot \hat{x} \\
& = pf(\tfrac{1}{2}(x^* + x^{**})) - w \cdot \tfrac{1}{2}(x^* + x^{**}) \\
& = pf(\tfrac{1}{2}(x^* + x^{**})) - \tfrac{1}{2}(w \cdot x^* + w \cdot x^{**}) \\
& > p\tfrac{1}{2}[f(x^*) + f(x^{**})] - \tfrac{1}{2}(w \cdot x^* + w \cdot x^{**}) \\
& = \pi^*.
\end{aligned}$$

Thus, $\hat{x}$ is a strictly better choice than x^* and x^{**}, contradicting the assumption that these are optimal choices.

Question 22. (Microeconomics 1, December 2016) According to the Lincoln Longwool Sheep Breeders Association, the Lincoln Longwool sheep is "one of the most important breeds ever seen in our green and pleasant land." It is a "dual-purpose" breed, meaning it yields high quality wool and meat. Suppose that sheep live for up to two years. If a sheep is killed at the end of the first year, it yields both wool and meat. If a sheep is killed at the end of the second year, it yields wool in both years and the same amount of meat. Households are endowed with sheep, and consume meat and wool each year. Households' preferences can be represented with a discounted utility function. Farms buy sheep to produce wool and meat.

(i) Write down a competitive model of the sheep, wool and meat markets across the two years.

Comment. Almost no students answered this question correctly. In particular, few students correctly accounted for dead and live sheep.

Answer.

Households. There are N households, which are endowed with s sheep. Let (w_t, m_t) be the household wool consumption and meat consumption in time period $t \in \{1, 2\}$, which gives the household utility

$$u(w_1, m_1) + \beta u(w_2, m_2).$$

The corresponding prices are (p_t^s, p_t^w, p_t^m). In partiular, p_t^s is the price of renting a sheep for period t. The household receives a share of the firm's profits, π/N. The household's utility maximisation problem is

$$\begin{aligned} \max_{(w_t, m_t)_{t\in\{1,2\}}} \quad & u(w_1, m_1) + \beta u(w_2, m_2) \\ \text{s.t. } & p_1^w w_1 + p_2^w w_2 + p_1^m m_1 + p_2^m m_2 = (p_1^s + p_2^s)s + \tfrac{\pi}{N} \end{aligned}$$

Farm. It allocates K_t (killed) sheep for meat and wool production, and L_t (live) sheep for wool production only, so that $M_t = f(K_t)$ meat is produced and $W_t = g(K_t + L_t)$ wool is produced. It needs to acquire $S_1 = K_1 + L_1$ sheep in year 1, and $S_2 = K_1 + K_2 + L_2$ in year 2. The farm's profit function is

$$\begin{aligned} & \pi((p_t^s, p_t^w, p_t^m)_{t\in\{1,2\}}) \\ & = \max_{K_1, K_2, L_1, L_2} p_1^m f(K_1) + p_2^m f(K_2) + p_1^w g(K_1 + L_1) + p_2^w g(K_2 + L_2) \\ & \qquad - p_1^s(K_1 + L_1) - p_2^s(K_1 + K_2 + L_2). \end{aligned}$$

Equilibrium. A price vector $(p_t^s, p_t^w, p_t^m)_{t\in\{1,2\}}$ and an allocation

$$(w_t, m_t, M_t, L_t)_{t\in\{1,2\}}$$

constitute an equilibrium if the allocation solves the choice problems above, and markets clear:

$$\begin{aligned} Ns &= K_1 + L_1 \\ Ns &= K_1 + K_2 + L_2 \\ Nm_1 &= f(K_1) \\ Nm_2 &= f(K_2) \\ Nw_1 &= g(K_1 + L_1) \\ Nw_2 &= g(K_2 + L_2). \end{aligned}$$

(ii) Prove that farms demand more sheep in the first year if the price of sheep decreases (but no other prices change).

Answer. By the envelope theorem,

$$\begin{aligned} &\frac{\partial \pi(p_1^s, p_1^m, p_1^w, p_2^s, p_2^m, p_2^w)}{\partial p_1^s} \\ &= \Bigg[\frac{\partial}{\partial p_1^s}\Bigg(p_1^m f(K_1) + p_2^m f(K_2) + p_1^w g(K_1 + L_1) + p_2^w g(K_2 + L_2) \\ &\qquad - p_1^s(K_1 + L_1) - p_2^s(K_1 + K_2 + L_2)\Bigg]_{\text{at optimal } L_1, K_1, L_2, K_2} \\ &= -\left[K_1 + L_1\right]_{\text{at optimal } L_1, K_1} \\ &= -S_1(p_1^s, p_1^m, p_1^w, p_2^s, p_2^m, p_2^w). \end{aligned}$$

Now, the profit function π is a convex function, since it is the upper envelope of linear functions (one linear function for each (K_1, L_1, K_2, L_2)). Therefore, $\frac{\partial \pi}{\partial p_1^s}$ is increasing in p_1^s. Since the left side is increasing, we conclude that sheep demand S_1 is a decreasing function of the price of sheep p_1^s.

(iii) Write down the firm's value of owning live sheep in the first and second years, making use of a Bellman equation. Prove that these are concave functions of the number of sheep.

Answer. The value of S_1 sheep in the first year is

$$\begin{aligned} V_1(S_1; p_1^w, p_1^m, p_2^w, p_2^m) = \max_{K_1, S_2} &\; p_1^m f(K_1) + p_1^w g(S_1) + V_2(S_2; p_2^w, p_2^m) \\ \text{s.t. } &\; K_1 + S_2 = S_1. \end{aligned}$$

The value of S_2 sheep in the second year is

$$V_2(S_2; p_2^w, p_2^m) = p_2^m f(S_2) + p_2^w g(S_2).$$

If we assume that the production functions f and g are concave in S_2, then V_2 is the sum of two concave functions, and hence is concave in S_2.

Similarly, the objective in the Bellman equation is jointly concave in (S_1, K_1, S_2) and the constraint is linear, so V_1 is concave in S_1 (by a Theorem from lectures).

(iv) Find an assumption on the model parameters such that the price of sheep decreases over time.

Answer. The firm's first-order conditions with respect to live-sheep are:

$$\begin{aligned} p_1^w g'(K_1 + L_1) &= p_1^s \\ p_2^w g'(K_2 + L_2) &= p_2^s. \end{aligned}$$

By the market clearing condition and looking at endowments, note that $K_2 + L_2 = L_1$.

The household's first-order conditions with respect to wool consumption are:

$$\begin{aligned} u_w(w_1, m_1) &= \lambda p_1^w \\ \beta u_w(w_2, m_2) &= \lambda p_2^w. \end{aligned}$$

where λ is the Lagrange multiplier on the budget constraint.

Substition gives:

$$\begin{aligned} \tfrac{1}{\lambda} u_w(w_1, m_1) g'(K_1 + L_1) &= p_1^s \\ \tfrac{1}{\lambda} \beta u_w(w_2, m_2) g'(L_1) &= p_2^s. \end{aligned}$$

Therefore $p_1^s > p_2^s$ if and only if

$$u_w(w_1, m_1) g'(K_1 + L_1) > \beta u_w(w_2, m_2) g'(L_1).$$

Now, if we assume that g has constant returns to scale, then $p_1^s > p_2^s$ is if

$$u_w(w_1, m_1) > \beta u_w(w_2, m_2).$$

If $\beta = 0$, then this inequality holds; it would also hold for small β if u_w is bounded. Specifically, if there are some number $x, y > 0$ such that $u_w(w, m) \in [x, y]$ for all (w, m), then the left side is bigger if $\beta < \frac{x}{y}$.

Conclusion: if β is close to zero, the marginal utility of wool is bounded, and the wool production function has constant returns to scale, then the price of sheep decreases over time.

(v) Suppose that half of the population is poor, and only has only half of the sheep endowment. Is it possible to devise a lump-sum transfer scheme that institutes equal welfare for each household?

Comment. You can answer either by citing the second welfare theorem, or by constructing the policy directly. If you apply the second welfare theorem, you have to prove that the target allocation is efficient (which can be done via the first welfare theorem).

Answer. Yes. Suppose the wealthy households are endowed with s sheep, and the poor households are endowed with $s/2$ sheep. Let $(\hat{p}_t^s, \hat{p}_t^m, \hat{p}_t^w)$ be the equilibrium prices once the tax regime is implemented. Taxing the wealthy households $(\hat{p}_1^s + \hat{p}_2^s)s/4$ and transferring this value to the poor households would give all households the same budget constraint. Therefore, all househoulds would have the same preferences and same budget constraint, so they would have the same welfare.

(vi) * Let $X = \mathbb{R}_+^6$. Suppose there is a continuous function $f : X \to X$ with the properties that (1) $p \in X$ is an equilibrium price vector if and only if $f(p) = p$ and (2) $f(tx) = f(x)$ for all $t > 0$. (a) Apply Brouwer's fixed point theorem to prove that an equilibrium exists. *Hint: you will need to reformulate f.* (b) Fix any $p_0 \in X$. Without using Brouwer's point theorem, prove that if the sequence $p_{n+1} = f(p_n)$ is a Cauchy sequence, then f has a fixed point.

Comment. The question on the exam incorrectly defined X as $\mathbb{R}_{++}^6$.

Answer.

(a) Let $Y = \left\{p \in X : \sum_{i=1}^N p_i = 1\right\}$ and

$$g(x) = \frac{f(p)}{\sum_{i=1}^N f_i(p)}.$$

Notice that $g : Y \to Y$ is continuous (since f and the standard operations + and / are continuous). Next, Y is closed, bounded, and convex, so Brouwer's fixed point theorem implies that g has a fixed point p^*. Now, let $t = \sum_{i=1}^N f_i(p^*)$. By construction, $f(p^*) = tp^*$. By property (2) $f(f(p^*)) = f(tp^*) = f(p^*)$. We conclude that $f(p^*)$ is a fixed point of f. By property (1) $f(p^*)$ is an equilibrium price vector.

(b) This is an important part of the proof of Banach's fixed point theorem. Since (X, d_2) is complete, p_n is a convergent sequence. Let p^* be its limit. Since f is continuous, $f(p_n) \to f(p^*)$. Since $p_{n+1} = f(p_n)$, the sequence

$f(p_n)$ is a subsequence of p_n, so $f(p_n) \to p^*$. Since $f(p_n)$ converges both to p^* and $f(p^*)$, we conclude that $p^* = f(p^*)$. Therefore, p^* is a fixed point of f.

Question 23. (Microeconomics 1, December 2016) Suppose a country consists of workers with and without university degrees. Only university graduates can design machines, but both types of worker are equally competent at operating machines. There are two firms: a machine manufacturer that hires university graduates and a clothing manufacturer that buys machines and can hire either type of worker. Workers sell labour and consume clothing and machines (for washing their clothes).

(i) Formulate a competitive equilibrium model of the markets for both types of labour, machines and clothing. *Hint: do not assume that equilibria are symmetric.*

Comment. The original formulation of the question did not make it clear that machines are necessary to make clothes. As a result, all students rightly did not include machines in the clothing production function – it is good to keep your model as simple as possible.

Answer.

Workers' problem. Let H be the set of workers, and let $G \subseteq H$ be the set of graduates. Let (e_g^h, e_n^h) be the endowment of worker $h \in H$ of graduate hours and non-graduate hours of labour. Assume that if $h \in G$ then $(e_g^h, e_n^h) = (1, 0)$; otherwise $(e_g^h, e_n^h) = (0, 1)$. Worker h supplies (l_g^h, l_n^h) units of labour at wages (w_g, w_n), and consumes c^h items of clothing at price p and m^h machines at price r. The worker receives a share of the firms' profits $\frac{\pi}{|H|}$, where $\pi = \pi^m(r; w_g) + \pi^c(p; w_g, w_n, r)$ is defined below. The worker's utility is $u(l_g^h + l_n^h, c^h, m^h)$, so the utility maximisation problem is

$$\max_{l_g^h, l_n^h, c^h, m^h} u(l_g^h + l_n^h, c^h, m^h)$$
$$\text{s.t. } pc^h + rm^h = w_g l_g^h + w_n l_n^h + \tfrac{\pi}{|H|} \text{ and } l_g^h \leq e_g^h \text{ and } l_n^h \leq e_n^h.$$

Machine manufacturer's problem. The machine manufacturer hires L_g^m graduates and sells $M^m = f(M^m)$ machines. Its profits are

$$\pi^m(r; w_g) = \max_{L_g^m} r f(L_g^m) - w_g L_g^m.$$

Clothing manufacturer's problem. The clothing manufacturer hires L_g^c graduates and L_n^c non-graduates, and buys M^c machines to sell $C^c = g(L_g^c + L_n^c, M^c)$ units of clothing. Its profits are

$$\pi^c(p; w_g, w_n, r) = \max_{L_g^c, L_n^c, M^c} pg(L_g^c + L_n^c, M^c) - w_g L_g^c - w_n L_n^c - rM^c.$$

Equilibrium. The prices (p, r, w_g, w_n) and allocation

$$\left(\left\{c^h, l_g^h, l_n^h, m^h\right\}_{h\in H}, L^m, M^m, C^c, L_g^c, L_n^c, M^c\right)$$

form an equilibrium if the choices solve the respective problems above, and each market clears:

$$\sum_{h\in H} l_g^h = L_g^m + L_g^c$$

$$\sum_{h\in H} l_n^h = L_n^c$$

$$\sum_{h\in H} c^h = C^c$$

$$\sum_{h\in H} m^h + M^c = M^m.$$

(ii) Suppose at some market prices, the supply of university-educated labour exceeds demand. Does this imply that the demand for uneducated labour exceeds supply?

Answer. While Walras' law implies that there must be excess supply in one market, it need not be in the uneducated labour market.

(iii) Suppose the two firms decide to merge into single firm. (a) Write the combined-firm's profit-maximisation problem using a Bellman equation to separate the output and input choices. (b) Does the equilibrium (or equilibria) change after the merger?

Comment. One common mistake in this part was to combine the graduate labour demand across the two activities (making machines and making clothes). Students who made this mistake effectively assumed that university graduates could spend the whole day doing both tasks simultaneously.

Another common mistake was to forget to include the wages as state variables.

Answer.

(a) The profit function is

$$\pi(p, r, w_g, w_n) = \max_{C,M} pC + rM - e(C, M; w_g, w_n)$$

where the production cost is

$$\begin{aligned} e(C, M; w_g, w_n) = &\min_{M^c, L_g^c, L_n^c, L_g^m} w_g(L_g^c + L_g^m) + w_n L_n^c \\ &\text{s.t. } f(L_g^m) \geq M + M^c \text{ and } g(L_g^c + L_n^c, M^c) \geq C. \end{aligned}$$

(b) No. The merged firm would make the same choices as the two separate firms if confronted with the same prices. Therefore, the set of equilibria would be unchanged.

(iv) Prove that if the wages of uneducated workers increases, the clothing manufacturer hires fewer uneducated workers.

Answer. By the envelope theorem,

$$\frac{\partial \pi^c(p; r, w_g, w_n))}{\partial w_n} = -L_n(p; r, w_g, w_n).$$

Now, π^c is convex, as it is the upper envelope of a set of linear functions (one for each input choice (M^c, L_g^c, L_n^c)). Therefore, the left side is increasing in w_n. We conclude that $L_n(p; r, w_g, w_n)$ is decreasing in w_n, i.e. when uneducated wages increase, the clothing manufacturer's demand for uneducated labour decreases.

(v) Prove that if the clothing manufacturer hires educated workers, then the wages paid to all workers by both firms are equal.

Comment. A common mistake was to differentiate the production function without specifying which partial derivative is relevant.

Many students did not assume that both types of workers are perfect substitutes in the clothing production function, which is important.

Many students confused the envelope theorem with first-order conditions. First-order conditions are about differentiating with respect to choice variables. The state variables of the profit function are all prices, which are not choice variables, so it makes no sense to apply the envelope theorem here. (The envelope theorem *is* useful for first-order conditions when the state variable is a choice, e.g. with production targets in the cost function.)

Answer. If the clothing manufacturer hires both types of worker, then the first-order conditions are:

$$L_g^c\colon\ pg_1(L_g^c + L_g^m, M^c) = w_g$$
$$L_n^c\colon\ pg_1(L_g^c + L_g^m, M^c) = w_n.$$

Since the left sides are equal, the right sides are equal and $w_g = w_n$.

(vi) Suppose every Pareto efficient allocation involves university graduates working for the machine manufacturer only. Is it possible to find lump-sum transfers to implement an allocation in which some university graduates work for the clothing manufacturer?

Comment. A common mistake here was to apply the second welfare theorem. But recall that this theorem only applies if the target allocation is efficient. The question implicitly states that the target allocation is inefficient.

Answer. No. By the first welfare theorem, every competitive equilibrium is efficient (no matter which endowments the households are allocated). Therefore, every competitive equilibrium with lump-sum transfers involves graduates working for the machine manufacturer only.

(vii) * Let $X = \mathbb{R}^4_+$. Suppose there is a continuous function $f : X \to X$ with the properties that (1) $p \in X$ is an equilibrium price vector if and only if $f(p) = p$ and (2) $f(tx) = f(x)$ for all $t > 0$. (a) Apply Brouwer's fixed point theorem to prove that an equilibrium exists. *Hint: you will need to reformulate* f. (b) Fix any $p_0 \in X$. Without using Brouwer's point theorem, prove that if the sequence $p_{n+1} = f(p_n)$ is a Cauchy sequence, then f has a fixed point.

Answer. See the answer to the last part of the previous question.

Question 24. (Advanced Mathematical Economics, May 2017 exam)
Part A
Until plastic bottles became popular in the 1960s, milk was sold in glass bottles that could be reused. For simplicity, assume there are two time periods. Suppose households supply labour, and buy bottled milk and empty bottles in both periods. Milk bottles from the first period become empty in the second period, and households can sell these (or buy even more). A firm uses labour to make bottles and bottled milk in both periods.

(i) Write down a competitive model of the bottled milk industry.

Comment. While most students described a self-consistent model, they did not capture the essence of the question:

- Milk producers make milk bottles out of empty bottles (and labour to make the milk).
- Households can sell their used bottles back to the milk producers.

Another common mistake was to only consider the total labour demand in each period, without thinking about how this labour is allocated between milk and bottle production. (For example, several students assumed that all workers conducted both activities simultaneously.)

Answer. We construct a representative **household** model with n identitical households. Let m_t be milk consumption, b_t be empty bottle usage, and h_t be labour supply in time period $t \in \{1, 2\}$. These trade at prices p_t, r_t and w_t respectively. The household is endowed with 1 unit of labour each period, and $1/n$ shares of the firm's profit Π (see below). The household maximises its utility function, $u(m_1, b_1, h_1) + \beta u(m_2, b_2, h_2)$. The household's problem is

$$\max_{(m_t, b_t, h_t)_{t\in\{1,2\}}} \quad u(m_1, b_1, h_1) + \beta u(m_2, b_2, h_2)$$
$$\text{s.t. } p_1 m_1 + p_2 m_2 + r_1 b_1 = r_2(m_1 + b_1 - b_2) + w_1 h_1 + w_2 h_2 + \Pi/n.$$

The **firm** allocates H_t^b and H_t^m units of labour to bottle and milk production in period t. The firm allocates B_t^m bottles for milk production in period t. The firm's bottle output is $B_t = f(H_t^b)$, and its milk output is $M_t = g(H_t^m, B_t^m)$

in period t. The firm's profit function is

$$\begin{aligned}
&\Pi(p_1, p_2, r_1, r_2, w_1, w_2) \\
&= \max_{H_1^b, H_2^b, H_1^m, H_2^m, B_1^m, B_2^m} \\
&\quad p_1 g(H_1^m, B_1^m) + p_2 g(H_2^m, B_2^m) + r_1[f(H_1^b) - B_1^m] + r_2[f(H_2^b) - B_2^m] \\
&\quad - w_1(H_1^m + H_1^b) - w_2(H_2^m + H_2^b)
\end{aligned}$$

An **equilibrium** consists of prices

$$(p_1, p_2, r_1, r_2, w_1, w_2)$$

and quantities

$$(m_1, m_2, h_1, h_2, b_1, b_2, H_1^b, H_2^b, H_1^m, H_2^m, B_1^m, B_2^m)$$

such that the choices solve the household and firm problems above, and all six markets clear:

$$nm_1 = M_1 \tag{56}$$
$$nm_2 = M_2 \tag{57}$$
$$nb_1 + B_1^m = B_1 \tag{58}$$
$$nb_2 + B_2^m = B_2 \tag{59}$$
$$nh_1 = H_1^m + H_1^b \tag{60}$$
$$nh_2 = H_2^m + H_2^b. \tag{61}$$

(ii) Reformulate the firm's problem by separating the firm's milk and bottle production decisions. *Hint: this is a bit like dynamic programming, but the "Bellman equation" has no choice variables.*

Answer. The firm could split into two firms, whose profit functions are related by the (trivial) Bellman equation

$$\Pi(p_1, p_2, r_1, r_2, w_1, w_2) = \Pi^m(p_1, p_2, r_1, r_2, w_1, w_2) + \Pi^b(r_1, r_2, w_1, w_2)$$

where

$$\begin{aligned}
&\Pi^m(p_1, p_2, r_1, r_2, w_1, w_2) \\
&= \max_{H_1^m, H_2^m, B_1^m, B_2^m} \\
&\quad p_1 g(H_1^m, B_1^m) + p_2 g(H_2^m, B_2^m) - r_1 B_1^m - r_2 B_2^m - w_1 H_1^m - w_2 H_2^m
\end{aligned}$$

and

$$\Pi^b(r_1, r_2, w_1, w_2) = \max_{H_1^b, H_2^b} r_1 f(H_1^b) + r_2 f(H_2^b) - w_1 H_1^b - w_2 H_2^b.$$

(iii) Prove that the firm has an increasing marginal profit (i.e. a decreasing marginal loss) of a second-period wage increase.

Answer. Π is the upper envelope of functions that are linear in prices (one for each choice vector). Therefore, Π is a convex function of prices, and hence a convex function of second-period wages. Therefore, its derivative with respect to second period wages is an increasing function.

(iv) Prove that the firm reacts to a second-period bottle price increase by increasing its net supply of (empty and filled) bottles.

Answer. By the envelope theorem,

$$\frac{\partial \Pi}{\partial r_2} = f(H_2^b(p_1, p_2, r_1, r_2, w_1, w_2)) - B_2^m(p_1, p_2, r_1, r_2, w_1, w_2). \tag{62}$$

The right side of this equation is the net supply of bottles. Since the left side is increasing in r_2 (see the previous question), the right side is also increasing.

Part B

Comment. Students made several common types of mistakes in this section:

- Students confused the meanings of "there exists" versus "for all".
- Students were confused about the definition of convergence. The phrase "for all $r > 0$, there exists an N such that" means something very different from "there exists an N such that for all $r > 0$".

(i) Consider the metric space (X, d_2) where $X = [0,1] \times \mathbb{R}$ and $d_2(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. What is the boundary of the set $A = [0,1] \times \{0\}$ in this space?

Answer. The boundary of A is A itself.

First we show that if $(u^*, v^*) \in A$, then $(u^*, v^*) \in \partial A$. Then the sequence $(u_n, v_n) = (u^*, 1/n) \notin A$ converges to (u^*, v^*). And the trivial sequence $(u_n', v_n') = (u^*, v^*) \in A$ also converges to (u^*, v^*). Therefore, $(u^*, v^*) \in \partial A$.

Second, we show that if $(u^*, v^*) \notin A$, then $(u^*, v^*) \notin \partial A$. Since $(u^*, v^*) \notin A$, we know that $v^* \neq 0$. Let $r = |v^*|$, where $r > 0$. Since the open ball $N_r(u^*, v^*)$ does not overlap with A, no sequence in A converges to (u^*, v^*).

(ii) Let $X = \{f : [0,1] \to \mathbb{R}$ s.t. f is continuously differentiable$\}$ and

$$d(f,g) = d_\infty(f,g) + d_\infty(f',g'),$$

where f' and g' are the derivatives of f and g respectively, and $d_\infty(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$. Prove (a) d is well-defined and (b) (X,d) is a metric space.

Comment. While it's possible to answer this question from first principles, the most elegant approach is to make use of the fact that $(CB([0,1]), d_\infty)$ is a metric space. (If you are worried that this is "cheating", you could prove this fact separately.)

Answer. (a) Checking that d is well-defined requires checking that d exists and is unique. Since $f,g \in X$, it follows that $f,g,f',g' : [0,1] \to \mathbb{R}$ are continuous functions. It follows that $x \mapsto |f(x)-g(x)|$ and $x \mapsto |f'(x)-g'(x)|$ are continuous functions. By the Weierstrass Theorem, the maxima of these continuous functions on the compact domain $[0,1]$ exist. Therefore, $d_\infty(f,g)$ and $d_\infty(f',g')$ exist. So d exists. Uniqueness is by construction, i.e. every step builds a unique item.

(b) We now prove that (X,d) is a metric space. We make use of the fact that (X,d_∞) is a metric space.

- $d(f,g) = 0$ if and only if $f = g$. Suppose $d(f,g) = 0$. Then $d_\infty(f,g) = 0$, and hence $f = g$.

  Suppose $f = g$. Then $f' = g'$. So $d_\infty(f,g) = 0$ and $d_\infty(f',g') = 0$. We conclude $d(f,g) = 0$.
- $d(f,g) = d(g,f)$. This follows from $d_\infty(f,g) = d_\infty(g,f)$ and $d_\infty(f',g') = d_\infty(g',f')$.
- $d(f,h) \le d(f,g) + d(g,h)$. Note that

$$d_\infty(f,h) \le d_\infty(f,g) + d_\infty(g,h), \tag{63}$$
$$d_\infty(f',h') \le d_\infty(f',g') + d_\infty(g',h'). \tag{64}$$

  Summing the two inequalities gives the conclusion.

(iii) Consider the metric space (X,d_1) where $X = (0,1)$ and $d_1(x,y) = |x-y|$. Supply a counter-example to prove that (X,d_1) is *not* complete.

Comment. Several students wrote "x_n wants to converge to x^*" without being aware of the limitations of using informal intuitive language rather than precise mathematical language. Intuitive language has its place in mathematical writing – it is very helpful for conveying difficult ideas (and this is why I speak this way in lectures). However, it is not a substitute for being precise; it should be used in addition to, not instead of precise language.

The problem with writing "wants to converge to" is it is terminology that has not been defined. It is probably possible to come up with a definition, but that would probably defeat the advantages of informal language. This means that the phrase is ambiguous. For example, suppose x_n is a sequence inside the metric space (X, d), but $x^* \notin X$. This means that x_n can not converge to x^*, because limits must lie inside X. The intuitive answer to this is that we can imagine a bigger metric space, (Y, d), that somehow extends (X, d). So "wanting to converge to x^*" means that x_n, when considered a sequence in (Y, d), converges to x^*. But this is still ambiguous, because there might be many different metric spaces (Y, d) that extend (X, d) in the right way.

To summarise: when you write proofs, you are very welcome to use intuitive and ambiguous language, *provided that you subsequently clarify exactly what you mean.*

Answer. We will construct a non-convergent Cauchy sequence. Let $x_n = 1/(n+1)$. In the metric space $([0,1], d_1)$, the sequence $x_n \to 0$. Therefore, x_n is a Cauchy sequence in (X, d_1), since the definition of Cauchy sequence is only based on the definition of the metric. Now, x_n does not converge, so (X, d_1) is not complete.

(iv) Consider the metric space (X, d_1), where $X \subseteq [0,1]$ and $d_1(x, y) = |x - y|$. Suppose that $x_n \in X$ has no convergent subsequence. Prove that X is not a closed set in $(\mathbb{R}, d_1)$.

Comment. Most students did not realise that compactness (i.e. the Bolzano-Weierstrass Theorem) is the key to this question. Compactness is the property that ensures that all sequences in $([0,1], d_1)$ have convergent subsequences. Because of this logical gap, students were quite creative in constructing specious arguments. For example, some students implicitly added the extra assumption that $x_n \to x^*$, where $x^* \in [0,1]$.

Answer. Suppose for the sake of contradiction that X were a closed set subset of $[0,1]$. Then X is a closed and bounded set, so the Bolzano-Weierstrass Theorem implies that $x_n \in X$ has a convergent subsequence. But x_n has no convergent subsequence. Therefore, the supposition that X is closed is false.

(v) Let $x_t \in [0,1]$ be the fraction of the population of generation t that is religious. Suppose that each subsequent generation's demographics are deterministic with $x_{t+1} = f(x_t)$, and that $x_t \to x^*$. Prove that if f is a continuous function, then x^* is a fixed point of f, i.e. x^* is a steady state.

Answer. Since f is continuous, $y_t = f(x_t)$ converges to $f(x^*)$. Now, y_t is a subsequence of x_t, so $y_t \to x^*$. Thus y_t converges to both x^* and $f(x^*)$. Since sequences can converge to only one point, we conclude that $x^* = f(x^*)$.

(vi) Prove that $f(x) = \frac{1}{3}x^2$ is a contraction on the metric space $(X, d) = ([0, 1], d_1)$ where $d_1(x, y) = |x - y|$.

Answer.

$$\begin{aligned} d_1(f(x), f(y)) &= \frac{1}{3}d_1(x^2, y^2) && (65) \\ &= \frac{1}{3}|x^2 - y^2| && (66) \\ &= \frac{1}{3}|(x - y)(x + y)| && (67) \\ &= \frac{1}{3}d_1(x, y)|x + y| && (68) \\ &\leq \frac{2}{3}d_1(x, y). && (69) \end{aligned}$$

Therefore, f is a contraction of degree $\frac{2}{3}$.

(vii) Consider a two player-game where player one and two choose $a \in [0, 1]$ and $b \in [0, 1]$ respectively. Suppose that player one and two have best response functions $f(b)$ and $g(a)$ respectively. Let $X = A \times B$ and $h : X \to X$ be defined by $h(a, b) = (f(b), g(a))$. Consider the following procedure (called iterated deletion of dominated strategies) for calculating Nash equilibria:

(a) Set $Y_1 = X$.

(b) Let $Y_{n+1} = h(Y_n)$, that is $Y_{n+1} = \{h(a, b) : (a, b) \in Y_n\}$.

(c) Report $Y_\infty = \cap_{n=1}^{\infty} Y_n$.

Prove that if h is continuous, then $Y_\infty \neq \emptyset$, i.e. that this procedure does not delete all strategies. *Hint: Apply the Cantor intersection theorem.*

Answer. We will show that each Y_n is compact, non-empty, and $Y_{n+1} \subseteq Y_n$. Then Cantor's intersection theorem establishes that Y_∞ is non-empty (and compact, but that is not relevant here).

Since h is continuous and Y_1 is non-empty and compact, it follows that $Y_2 = h(Y_1)$ is non-empty and compact. Repeating this argument establishes that each Y_n is non-empty and compact.

By assumption, $h(Y_1) \subseteq Y_1$, since best-responses must lie in X. Therefore, $h(h(Y_1)) \subseteq h(Y_1)$, and hence $Y_3 \subseteq Y_2$. Continuing this logic establishes that each $Y_{n+1} \subseteq Y_n$.

Therefore, Cantor's theorem applies.

(viii) Recall that $CB(\mathbb{R}_+)$ is the set of continuous and bounded functions with domain $\mathbb{R}_+$ and co-domain $\mathbb{R}$, whose distances can be measured with the metric

$$d_\infty(f,g) = \sup_{x\in\mathbb{R}_+} |f(x) - f(y)|.$$

Consider the following Bellman operator $\Phi : CB(\mathbb{R}_+) \to CB(\mathbb{R}_+)$, which is a contraction of degree β on $(CB(\mathbb{R}_+), d_\infty)$:

$$\begin{aligned}\Phi(V)(k) =& \max_{c,k'} u(c) + \beta V(k') \\ & \text{s.t. } c + k' = g(k).\end{aligned}$$

(You may interpret c as consumption, k as capital, $g(k)$ as output $u(c)$ as utility, and β as the rate of time preference.) Use Banach's fixed point theorem to prove that if u and g are concave, then the fixed point of Φ is concave.

Answer. We may reformulate Φ without the constraint as

$$\Phi(V)(k) = \max_{k'} u(g(k) - k') + \beta V(k').$$

We now show that if V is a concave function, then $\Phi(V)$ is also concave function. Let $h(k,k') = u(g(k)-k') + \beta V(k')$ be the objective function. Since u, g, $k' \mapsto -k'$ and V are concave functions it follows that h is a concave function. Now, let k'_1 and k'_2 maximise $h(k_1,\cdot)$ and $h(k_2,\cdot)$ respectively. Then,

$$\begin{aligned}\Phi(V)(tk_1 + (1-t)k_2) &= \max_{k'} h(tk_1 + (1-t)k_2, k') &(70)\\ &\geq h(tk_1 + (1-t)k_2, tk'_1 + (1-t)k'_2) &(71)\\ &\geq th(k_1,k'_1) + (1-t)h(k_2,k'_2) &(72)\\ &= t\Phi(V)(k_1) + (1-t)\Phi(V)(k_2). &(73)\end{aligned}$$

Thus, $\Phi(V)$ is concave whenever V is concave.

Let $X = \{f \in CB(\mathbb{R}) : f \text{ is concave}\}$. We have established that if we restrict Φ to X then Φ is a contraction in the metric space (X, d_∞). In tutorials, we established that (X, d_∞) is a complete metric space. Therefore Banach's fixed point theorem establishes that Φ has a unique fixed point $V^* \in X$, i.e. there is a fixed-point of Φ that is concave.

Similarly, Banach's fixed point theorem establishes that Φ has only one fixed point in $CB(\mathbb{R}_+)$, so the only fixed point must be V^*. We conclude that the fixed point of Φ is concave.

Question 25. (Microeconomics 1, May 2017) Consider an economy with two time-periods, in which the entire population lives for both periods. The young and old are identical, except the young have no labour endowment in the first period. They can supply up to their labour endowment and consume food in each period, and have time-separable preferences. A farm produces food from labour.

(i) Devise a competitive model of the food and labour markets.

Answer. Households. Each households $h \in H$ belongs to either generation $h \in X$ or $h \in Y$. In period 1, each old household $h \in X$ has a first-period labour endowment of $e_{h1} = 1$; the young households' $h \in Y$ have $e_{h1} = 0$. Apart from that households are identical. They all receive a share of the farm's profits π (see below). Second-period labour endowments are $e_{h2} = 1$, and each household chooses food consumption c_{ht}, labour supply $\ell_{ht} \in [0, e_{ht}]$, which trade at market prices p_t and w_t in period $t \in \{1, 2\}$ respectively. Each household's utility is

$$u_1(c_{h1}, \ell_{h1}) + \beta u_2(c_{h2}, \ell_{h2}).$$

Household h's utility maximisation problem is

$$\max_{(\ell_{ht} \in [0, e_{ht}], c_{ht} \in \mathbb{R}_+)_{t=1}^2} u_1(c_{h1}, \ell_{h1}) + \beta u_2(c_{h2}, \ell_{h2}) \tag{74}$$

$$\text{s.t. } p_1 c_{h1} + p_2 c_{h2} = w_1 \ell_{h1} + w_2 \ell_{h2} + \tfrac{\pi}{|H|}. \tag{75}$$

Farm. The farm uses labour L_t in period t to produce $Y_t = f(L_t)$ units of food. The farm's profit function is

$$\pi(p_1, p_2, w_1, w_2) = \max_{L_1, L_2} p_1 f(L_1) + p_2 f(L_2) - w_1 L_1 - w_2 L_2. \tag{76}$$

Equilibrium. An equilibrium consists of prices (p_1, p_2, w_1, w_2) and quantities

$$\left(\{c_{ht}, \ell_{ht}\}_{(h,t) \in H \times \{1,2\}}, L_1, L_2, Y_1, Y_2\right)$$

such that the quantities solve the respective problems above, and all four markets clear:

$$\sum_{h \in H} \ell_{h1} = L_1 \tag{77}$$

$$\sum_{h \in H} \ell_{h2} = L_2 \tag{78}$$

$$\sum_{h \in H} c_{h1} = Y_1 \tag{79}$$

$$\sum_{h \in H} c_{h2} = Y_2. \tag{80}$$

(ii) Suppose that at the (non-equilibrium) market prices, the market values of the excess demands for food sum to a positive number. Prove that there is excess supply in at least one of the labour markets. *Note: do not assume that there is excess demand in both food markets.*

Answer. One form of Walras' law is that for every vector of prices (p_1, p_2, w_1, w_2),

$$\sum_{h \in H} [p_1 c_{h1} + p_2 c_{h2} - w_1 \ell_{h1} - w_2 \ell_{h2}] - p_1 Y_1 - p_2 Y_2 + w_1 L_1 + w_2 L_2 = 0. \quad (81)$$

(This is obtained by summing all households' budget constraints and substituting in the farm's profits π.) Now, if the market value of the excess demand for food is

$$\sum_{h \in H} [p_1 c_{h1} + p_2 c_{h2}] - p_1 Y_1 - p_2 Y_2 \quad (82)$$

is positive, then the remaining terms (the market value of the excess demand for labour)

$$w_1 L_1 + w_2 L_2 - \sum_{h \in H} [w_1 \ell_{h1} + w_2 \ell_{h2}] \quad (83)$$

must be negative. This means there must be excess supply of labour in at least one time period.

(iii) Prove that the farm reacts to second-period food-price increases by increasing supply.

Answer. By the envelope theorem,

$$\frac{\partial \pi(p_1, p_2, w_1, w_2)}{\partial p_2} \quad (84)$$

$$= \frac{\partial}{\partial p_2} [p_1 f(L_1) + p_2 f(L_2) - w_1 L_1 - w_2 L_2] \Big|_{L_1 = L_1(p_1, p_2, w_1, w_2), L_2 = L_2(p_1, p_2, w_1, w_2)} \quad (85)$$

$$= f(L_2)|_{L_2 = L_2(p_1, p_2, w_1, w_2)} \quad (86)$$

$$= Y_2(p_1, p_2, w_1, w_2). \quad (87)$$

Since π is the upper envelope of a set of linear functions of prices (one for each (L_1, L_2) choice), π is a convex function. Therefore, the left side is increasing in p_2, so the right side is also increasing in p_2. We conclude that the firm's second-period food supply is increasing in the second period food price.

(iv) Write down the utility maximisation problem of a "big family" household that makes all market transactions on behalf of the households and the farm. Assume that the big-family household puts equal utility weight on all actual households. *Hints. Recall the home-production example from class. Put the market transactions in one Bellman equation, put the farm choices inside another Bellman equation, and bury the allocation of resources to households inside a value function.*

Answer. Upper case letters (L_1, etc.) are quantities that are traded, hatted letters ($\hat{L}_1$, etc.) are production quantities, and lower case letters (ℓ_{h1}, etc.) are household quantities. The big household's problem is:

$$\begin{aligned} &\max_{L_1,L_2,Y_1,Y_2} U(L_1, L_2, Y_1, Y_2) \\ &\text{s.t. } p_1 Y_1 + p_2 Y_2 = w_1 L_1 + w_2 L_2 \end{aligned}$$

where

$$\begin{aligned} U(L_1, L_2, Y_1, Y_2) = &\max_{\hat{L}_1,\hat{L}_2,\hat{Y}_1,\hat{Y}_2} V(\hat{L}_1, \hat{L}_2, \hat{Y}_1, \hat{Y}_2) \\ &\text{s.t.} \quad f(\hat{L}_1 - L_1) + Y_1 = \hat{Y}_1 \\ &\qquad\quad f(\hat{L}_2 - L_2) + Y_2 = \hat{Y}_2 \end{aligned}$$

and

$$\begin{aligned} V(\hat{L}_1, \hat{L}_2, \hat{Y}_1, \hat{Y}_2) = &\max_{\ell_{ht}\in[0,e_{ht}],c_{ht}\in\mathbb{R}_+} \sum_{h\in H} [u_1(c_{h1}, \ell_{h1}) + \beta u_2(c_{h2}, \ell_{h2})] \\ &\text{s.t.} \quad \sum_{h\in H} \ell_{h1} = \hat{L}_1 \\ &\qquad\quad \sum_{h\in H} \ell_{h2} = \hat{L}_2 \\ &\qquad\quad \sum_{h\in H} c_{h1} = \hat{Y}_1 \\ &\qquad\quad \sum_{h\in H} c_{h2} = \hat{Y}_2. \end{aligned}$$

(v) Suppose the government forcibly reallocated all resources to an efficient egalitarian allocation. If the population were allowed to trade based on their new endowments, what competitive allocation would arise?

Answer. No trade would be an equilibrium, i.e. the egalitarian allocation would become a competitive allocation. This is essentially the second welfare theorem, whose proof can be adapted to this situation as follows.

An equilibrium based on the new (egalitarian) endowments exists. (By the previous question, the economy can be reformulated as a pure-exchange equilibrium with a single household, and the pure-exchange existence theorem would apply.) Since the egalitarian allocation is the endowment, no household can be worse off under the new equilibrium allocation. Since the egalitarian allocation is efficient, this means that no household can be strictly better off. Therefore, every household is indifferent between the egalitarian allocation and the new equilibrium allocation. Thus, under the new equilibrium prices, the egalitarian allocation is an equilibrium allocation.

(vi) * Give an example of a metric space with the property that every closed subset is compact.

Answer. Any compact metric space, for example $([0,1], d_2)$. This is because in every metric space (X, d), every closed subset A of a compact set K is compact. We proved this in class:

Let $a_n \in A$ be any sequence. Since $a_n \in K$ and K is compact, a_n has a convergent subsequence $b_n \to b$ with $b \in K$. Since A is closed, $b \in A$. We conclude that A is compact.

(vii) * Prove the Cantor intersection theorem:

Let (X, d) be a metric space. Suppose $A_n \subseteq X$ is a sequence of sets such that (a) $A_{n+1} \subseteq A_n$, (b) $A_n \neq \emptyset$ and (c) A_n is compact for all n. Let $A = \cap A_n$. Then $A \neq \emptyset$.

Answer. See the proof of Cantor's intersection theorem in the notes.

Question 26. (Microeconomics 1, May 2017) A café and a restaurant both serve meals to customers, using labour and food. The restaurant requires double the labour and food inputs to produce the same number of meals as the café. Households supply labour and only eat at restaurantes and/or cafes. At every level of consumption and supply, households prefer an extra restaurant meal to an extra café meal. A farm produces food from labour only.

(i) Write down a competitive equilibrium model of the labour, food, and meals (restaurants and cafes) markets.

Answer.

Households choose restaurant and café meals m^r and m^c and work hours h at prices p^r, p^c, w respectively to maximise utility $u(m^r, m^c, h)$. There are n households, and each household receives an equal share of all firms' profits Π. The representative household's utility maximisation problem is:

$$\max_{m^r, m^c, h} u(m^r, m^c, h) \tag{88}$$

$$\text{s.t. } p^r m^r + p^c m^c = wh + \Pi/n. \tag{89}$$

Firms. There are three firms, with total profits $\Pi = \pi^f + \pi^c + \pi^r$ arising from the farm, café, and restaurant, respectively. The **farm** uses H^f hours of labour to produce $Y = f(H^f)$ units of food which it sells at price p^y. Its profit function is

$$\pi^f(p^y, w) = \max_{H^f} p^y f(H^f) - wH^f. \tag{90}$$

The **café** uses H^c hours of labour and y^c units of food to produce $M^c = g(H^c, y^c)$ café meals. Its profit function is

$$\pi^c(p^c, p^y, w) = \max_{H^c, y^c} p^c g(H^c, y^c) - wH^c - p^y y^c. \tag{91}$$

The **restaurant** uses H^r hours of labour and y^r units of food to produce $M^r = g(H^r/2, y^r/2)$ restaurant meals. Its profit function is

$$\pi^r(p^r, p^y, w) = \max_{H^r, y^r} p^r g(H^r/2, y^r/2) - wH^r - p^y y^r. \tag{92}$$

An **equilibrium** consists of prices (p^y, p^c, p^r, w) and quantities

$$(m^r, m^c, h, H^f, H^c, H^r, M^c, M^r, Y, y^c, y^r)$$

such that the quantities solve the respective problems above given these prices, and all four markets clear:

$$y^c + y^r = Y \tag{93}$$

$$nm^c = M^c \tag{94}$$

$$nm^r = M^r \tag{95}$$

$$nh = H^c + H^r + H^f. \tag{96}$$

(ii) Suppose there is an equilibrium in which restaurant meals cost £1. Does this mean that there is an equilibrium in which café meals cost £1?

Comment. Most students misinterpreted the question as asking: is there an equilibrium in which both types of meals cost £1? The first part of the question consists of almost irrelevant information. However, it is not completely irrelevant: it rules out the possibility that there is no equilibrium at all.

Answer. Yes. Let $p = (p^y, p^c, p^r, w)$ be the prices in the equilibrium for which restaurant meals cost $p^r = 1$. Then the price vector p/p^c combined with the same quantities give an equilibrium in which café meals cost 1.

(iii) Prove that in every equilibrium in which café meals are sold, restaurant meals trade at a higher price than café meals.

Answer. If café meals were more or equally expensive, then households would not buy them.

(iv) Prove that the marginal cost of restaurant meals equals the wage divided by the marginal productivity of labour.

Answer. There are at least two approaches: (1) construct the cost function of a lazy manager who demands the same amount of food regardless of the meal production target or price, or (2) apply the Lagrange-multiplier version of the envelope theorem to the cost function, and calculate the relevant multipliers using first-order conditions. All correct answers from students used the second approach. I pursue the first option.

The restaurant's profit function can be decomposed into input and output choices:

$$\pi^r(p^r, p^y, w) = \max_{M^r} p^r M^r - C^r(M^r; p^y, w) \tag{97}$$

where

$$C^r(M^r; p^y, w) = \min_{H^r, y^r} wH^r + p^y y^r \tag{98}$$

$$\text{s.t. } g(H^r/2, y^r/2) \geq M^r. \tag{99}$$

Suppose that $(\bar{M}^r; \bar{p}^y, \bar{w})$ are the equilibrium production target and prices, and that the best choices are $(\bar{H}^r, \bar{y}^r)$. Now, consider a lazy manager who only adjusts the labour demand in response to a production target or price change. His value function is:

$$L^r(M^r; p^y, w) = wH^r(M^r; \bar{M}^r, \bar{p}^y, \bar{w}) + p^y \bar{y}^r, \tag{100}$$

where $H^r(M^r; \bar{y}^r)$ is the lazy demand function defined implicitly the equation

$$g(H^r/2, \bar{y}^r/2) = M^r. \tag{101}$$

By the implicit function theorem, the lazy manager's marginal cost is

$$\begin{aligned}
\frac{\partial L^r(M^r; p^y, w)}{\partial M^r} &= w \frac{\partial H^r(M^r; \bar{M}^r, \bar{p}^y, \bar{w})}{\partial M^r} && (102)\\
&= -w \frac{-1}{g_1(H^r(M^r; \bar{y}^r)/2, \bar{y}^r/2)} && (103)\\
&= \frac{w}{g_1(H^r(M^r; \bar{y}^r)/2, \bar{y}^r/2)}. && (104)
\end{aligned}$$

Since the lazy value function is tangent to the cost function (e.g. due to the differentiable sandwich lemma), we have

$$\begin{aligned}
&\left.\frac{\partial C^r(M^r; p^y, w)}{\partial M^r}\right|_{(M^r; p^y, w) = (\bar{M}^r; \bar{p}^y, \bar{w})} && (105)\\
&= \left.\frac{\partial L^r(M^r; p^y, w)}{\partial M^r}\right|_{(M^r; p^y, w) = (\bar{M}^r; \bar{p}^y, \bar{w})} && (106)\\
&= \frac{\bar{w}}{g_1(\bar{H}^r/2, \bar{y}^r/2)}. && (107)
\end{aligned}$$

(v) Prove that the restaurant's marginal cost curve is increasing.

Answer. The marginal cost curve is given by

$$\begin{aligned}
C^r(M^r; p^y, w) = \min_{H^r, y^r} \; & wH^r + p^y y^r && (108)\\
\text{s.t. } & g(H^r/2, y^r/2) \geq M^r. && (109)
\end{aligned}$$

We check that it is convex, based on the assumption that g is a concave production function. Let (H, y) and (H', y') be optimal plans to meet production targets M and M' respectively. Then

$$\begin{aligned}
tC^r(M; p^y, w) + (1-t)C^r(M'; p^y, w) &= t(wH + p^y y) + (1-t)(wH' + p^y y') \quad (110) \\
&= w(tH + (1-t)H') + p^y(ty + (1-t)y) \quad (111) \\
&\geq C^r(tM + (1-t)M'; p^y, w). \quad (112)
\end{aligned}$$

The last step is true because the intermediate plan $t(H, y) + (1-t)(H', y')$ meets the intermediate target $tM+(1-t)M'$, but is not necessarily the lowest cost plan to do so. This is because g is a concave production function.

(vi) The goverment would like to increase restaurant meal consumption. It proposes (symmetric) lump-sum transfer scheme from households to the restaurant. Would this policy have the desired effect?

Comment. Answers to this question were often misleading and/or incomplete. For example, one student wrote:[1]

> No, under the first welfare theorem, the equilibrium is already efficient and no lump-sum tax transfer under the same feasibility constraints can lead to a Pareto improvement.

These two statements are true, but incomplete. In particular, they do not rule out the possibility that policy might increase restaurant meal consumption either (i) by delivering an inefficient allocation, (ii) by making someone better off and someone else worse off, or (iii) by having no effect on welfare at all.

Answer. No, it would have no effect at all. Since the households hold equal shares in the restaurant, each household would have a net tax of zero. Adding a constant to the firm's objective does not affect its optimal choices.

(vii) * Give an example of a metric space with the property that every closed subset is compact.

Answer. This is a repeat from the previous question.

(viii) * Prove the Cantor intersection theorem:

Let (X, d) be a metric space. Suppose $A_n \subseteq X$ is a sequence of sets such that (a) $A_{n+1} \subseteq A_n$, (b) $A_n \neq \emptyset$ and (c) A_n is compact for all n. Let $A = \cap A_n$. Then $A \neq \emptyset$.

Answer. See the proof of Cantor's intersection theorem in the notes.

[1] I simplified the English a little bit.

Question 27. (Advanced Mathematical Economics, December 2017) **Part A**

Internet data centres generate waste energy that can be used to heat homes. Suppose that this waste energy can be transported but not stored. Households benefit more from heat during the evening, and benefit more from the Internet during the day. Households own the data centres, which they rent out to the internet company.

(i) Write down a competitive equilibrium model of the data centre, computing and heat markets during the day and evening.

Comment. Most students answered this part well. Most of the mistakes were items from the checklist (see the start of this document) and did not relate specifically to the structure of this question.

Answer. Households. There are n identical households, and two time periods ($t = 1$ is day and $t = 2$ is night). Each household is endowed with equipment e which they rent at prices r_t, and buy heat h_t metered at price m_t, and internet services i_t at price p_t. These choices give them utility $u_1(h_1, i_1) + u_2(h_2, i_2)$. Each household receives a dividend π/n. The utility maximisation problem is

$$\begin{aligned} &\max_{h_1,i_1,h_2,i_2} u_1(h_1, i_1) + u_2(h_2, i_2) \\ &\text{s.t. } m_1h_1 + m_2h_2 + p_1i_1 + p_2i_2 = r_1e + r_2e + \pi/n. \end{aligned}$$

Internet firm. The internet firm buys equipment E_t and produces heat $H_t(E_t)$ and internet services $I_t(E_t)$ in time t. Its profit function is

$$\begin{aligned} &\pi(r_1, r_2, m_1, m_2, p_1, p_2) \\ &= \max_{E_1,E_2} m_1H_1(E_1) + m_2H_1(E_2) + p_1I_1(E_1) + p_2I_2(E_2) - r_1E_1 - r_2E_2. \end{aligned}$$

Equilibrium. An equilibrium consists of prices $(r_1, r_2, m_1, m_2, p_1, p_2)$ and quantities $(E_1, E_2, h_1, i_1, h_2, i_2)$ such that all choices solve the problems above, and all markets clear:

$$\begin{aligned} ne &= E_1 \\ ne &= E_2 \\ nh_1 &= H_1(E_1) \\ nh_2 &= H_2(E_2) \\ ni_1 &= I_1(E_1) \\ ni_2 &= I_2(E_2). \end{aligned}$$

(ii) Prove that if the daytime heating price increases, then the firm sells more daytime internet services.

Comment. Most students answered this question well.

Answer. By the envelope theorem,

$$\begin{aligned}
&\frac{\partial \pi(r_1, r_2, m_1, m_2, p_1, p_2)}{\partial m_1} \\
&= \left[\frac{\partial}{\partial m_1}(m_1 H_1(E_1) + m_2 H_1(E_2) + p_1 I_1(E_1) + p_2 I_2(E_2) - r_1 E_1 - r_2 E_2)\right]_{\text{at optimal }(E_1, E_2)} \\
&= [H_1(E_1)]_{\text{at optimal } E_1} \\
&= H_1(E_1(r_1, r_2, m_1, m_2, p_1, p_2)).
\end{aligned}$$

Now, since the profit function π is the upper envelope of linear functions (one function for each input choice (E_1, E_2)), we conclude that π is a convex function. Therefore the left side is increasing in m_1. So the right side is increasing in m_1.

We conclude that the demand for equipment E_1 would increase, and hence internet services $I_1(E_1)$ would increase.

(iii) Write down a Bellman equation that separates the household's problem into day and evening choices.

Comment. Most students struggled with this question. We didn't go into much detail for this type of dynamic programming in class this year, but there are many practice questions like this.

Answer. Let a be assets saved from the first period. Then the utility maximisation problem can be decomposed into:

$$\begin{aligned}
v_1(r_1, r_2, m_1, m_2, p_1, p_2) = \max_{h_1, i_1, a} \; & u_1(h_1, i_1) + v_2(a; r_2, m_2, p_2) \\
\text{s.t. } & m_1 h_1 + p_1 i_1 + a = r_1 e,
\end{aligned}$$

where

$$\begin{aligned}
v_2(a; r_2, m_2, p_2) = \max_{h_2, i_2} \; & u_2(h_2, i_2) \\
\text{s.t. } & m_2 h_2 + p_2 i - 2 = r_2 e + a.
\end{aligned}$$

Part B

Comment. Overall, most students answered many questions, but answered few questions well. In other words, most students need to improve the way they write their proofs.

(i) Let (X, d_X) and (Y, d_Y) be complete metric spaces. Let $Z = X \times Y$ and $d_Z((x,y),(x',y')) = \max\{d_X(x,x'), d_Y(y,y')\}$. Prove that (Z, d_z) is a complete metric space.

Comment. Most students struggled with this question. While many students seemed to have good intuition about what this question was about, they could not translate this into a logical explanation. The most common mistake was to start by assuming that x_n and y_n are Cauchy sequences. The proof needs to start by assuming that z_n is a Cauchy sequence. This *implies* that x_n and y_n are Cauchy sequences (but this requires a proof).

Answer. Let $z_n \in Z$ be a Cauchy sequence. We need to prove that z_n is convergent.

Since $z_n = (x_n, y_n)$ is a Cauchy sequence, it follows that x_n is a Cauchy sequence. Specifically, pick any $r > 0$. Then there exists some N such that:

- $d_Z(x_n, y_n; x_m, y_m) < r$ for all $n, m > N$, and hence
- $\max\{d_X(x_n, x_m), d_Y(y_n, y_m)\} < r$ for all $n, m > N$, and hence
- $d_X(x_n, x_m) < r$ for all $n, m > N$.

Since x_n is a Cauchy sequence inside the complete metric space (X, d_X), we conclude that x_n is convergent, i.e. $x_n \to x^*$ for some $x^* \in X$. By similar reasoning, $y_n \to y^*$ for some $y^* \in Y$.

It remains to show that $(x_n, y_n) \to (x^*, y^*)$. Pick any $r > 0$. Since $x_n \to x^*$, there exists some N_x such that for all $n > N_x$, $d_X(x_n, x^*) < r$. Since $y_n \to y^*$, there exists some N_y such that for all $n > N_y$, $d_Y(y_n, y^*) < r$. Therefore, for all $n > N = \max\{N_x, N_y\}$,

$$d_z(x_n, y_n; x^*, y^*) = \max\{d_X(x_n, x^*), d_Y(y_n, y^*)\} < r.$$

(ii) Let (X, d) be any metric space, and $A \subseteq X$ any subset. Provide a counter-example to the following false statement: the interior of the boundary of A is empty, i.e. $\mathrm{int}(\partial A) = \emptyset$.

Comment. No student answered this question correctly. To make the question a bit easier, I could have added a hint: "think about incomplete metric spaces".

Answer. Consider the metric space $(\mathbb{R}, d_2)$ and the subset $A = \mathbb{Q}$. Then $\partial A = \mathbb{R}$, and $\mathrm{int}(\partial A) = \mathbb{R}$.

We verify that $\partial A = \mathbb{R}$. First, if $x \in \mathbb{R}\backslash A$ (i.e. x is irrational), then

- there is a sequence $a_n \in A$ with $a_n \to x$ (based on the decimal expansion of x, and
- the trivial sequence $b_n = x \in \mathbb{R}\backslash A$ has $b_n \to x$.

So in this case, $x \in \partial A$.

Second, if $x \in A$ (i.e. x is rational), then

- the trivial sequence $a_n = x \in A$ has $a_n \to x$, and
- the sequence $b_n = x + \frac{\sqrt{2}}{n} \in \mathbb{R}\backslash A$ and $b_n \to x$.

(iii) Prove that if x is a boundary point of A in (X, d) (defined in terms of sequences), then every open neighbourhood U of x has $U \cap A \neq \emptyset$ and $U \cap (X\backslash A) \neq \emptyset$.

Comment. Most students did not know what "open neighbourhood" means, incorrectly thinking it meant an open ball containing x. But this did not matter much, because any open neighbourhood of x would contain an open ball of radius r centred at x.

Most students failed to connect the radius of the open ball, r, with the definition of convergence of sequences.

Answer. Suppose $x \in \partial A$, and pick any open neighbourhood U of x.

Since $x \in \partial A$, there exists sequences $a_n \in A$ and $b_n \notin A$ such that $a_n \to x$ and $b_n \to x$. Since x is in the interior of U, there is an open ball $N_r(x) \subseteq U$.

Now, because $a_n \to x$, it follows that there exists some N_a such that for all $n \geq N_a$, $d(a_n, x) < r$. Similarly, there is some N_b such that for all $n \geq N_b$, $d(b_n, x) < r$.

We conclude that $a_{N_a} \in U \cap A$ and $b_{N_b} \in U \cap (X\backslash A)$.

(iv) Prove that $f : X \to Y$ is continuous if and only if for every open ball $U = N_r(y)$, the inverse image $f^{-1}(U)$ is an open set.

Comment. Most students adapted the open ball characterisation of continuity from class. This is fine, but I think it's easier to work with the open set characterisation of continuity.

Answer. We will make use of the following theorem: f is continuous if and only if for every open set $U \subseteq Y$, $f^{-1}(U)$ is an open set.

Since f is continuous, and pick any open ball $U = N_r(y)$. Since U is an open set, the theorem implies that $f^{-1}(U)$ is an open set.

Now suppose that $f^{-1}(U)$ is open for every open ball $U = N_r(y)$. We most prove that f is continuous. Let $V \subseteq Y$ be any open set. By the theorem, it suffices to show that $f^{-1}(V)$ is an open set. Since V is an open set, every point $v \in V$ is an interior point. Therefore, there exists an open ball $B_v = N_{r_v}(v) \subseteq V$ for every $v \in V$. Notice that $V = \cup_{v\in V} B_v$. Therefore

$$f^{-1}(V) = \cup_{v\in V} f^{-1}(B_v).$$

By the condition, each $f^{-1}(B_v)$ is an open set. Moreover, the union of open sets is open. We conclude that $f^{-1}(V)$ is an open set.

(v) Suppose $f : \mathbb{R} \to \mathbb{R}$, where distances in both the domain and co-domain are measured with the Euclidean metric. Suppose that $\lim_{n\to\infty} f(1/n) = 1$ and $f(0) = 0$. Provide an example of an open set U such that $f^{-1}(U)$ is not open.

Comment. No student answered this question correctly.

Answer. Let $U = N_{0.5}(0) = (-\frac{1}{2}, \frac{1}{2})$ and $V = f^{-1}(U)$. Since U is an open ball, it is an open set. We will show that V is not an open set, contradicting the open set characterisation of continuity.

Observe that $0 \in V$, since $f(0) = 0$. We will find a sequence $z_n \notin V$ such that $z_n \to 0$, and we will conclude that $0 \in \partial V$ and hence that V is not an open set.

Let $x_n = 1/n$ and $y_n = f(1/n)$. Since $y_n \to 1$, there is a number N such that $d_2(y_n, 1) < 1$ for all $n > N$. Let $z_n = x_{n+N}$. By construction, $z_n \notin V$. Since z_n is a subsequence of x_n, it follows that $z_n \to 0$.

(vi) Let x_t be the fraction of women that work in professional occupations. Assume that this changes over time according to $x_{t+1} = f(x_t)$ where $f : [0, 1] \to [0, 1]$, and distances are measured by the Euclidean metric. Now, suppose that (i) f is continuous, and that (ii) f is a contraction on $[0, \frac{1}{3})$, and also on $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, 1]$. Prove that there are either two or three steady-states (i.e. fixed points of f).

Comment. The most common mistakes in this question were:

- Applying Banach's fixed point theorem on $[0, \frac{1}{3})$, which is impossible since this is an incomplete metric space.
- Not accounting for the possibility that fixed points could lie at $\frac{1}{3}$ or $\frac{2}{3}$.

Answer. Suppose f is a contraction of degree a on these three sets.

First, we claim that f is a contraction on $[0, \frac{1}{3}]$. Recall from a homework problem that every distance metric is continuous, i.e. if $x_n \to x^*$ then $d(x_n, x_0) \to d(x^*, x_0)$. Let $x_n = \frac{1}{3} - \frac{1}{n}$. Since f is a contraction on $[0, \frac{1}{3})$, we know that for all $y \in [0, \frac{1}{3})$,

$$d(f(x_n), f(y)) \leq a d(x_n, y),$$

or equivalently, $g(x_n) \leq 0$ where

$$g(x) = d(f(x), f(y)) - ad(x, y).$$

Now, since g is continuous, it follows that $g(\frac{1}{3}) \leq 0$, and hence

$$d(f(\tfrac{1}{3}), f(y)) \leq ad(\tfrac{1}{3}, y).$$

So f is a contraction on $[0, \frac{1}{3}]$.

Similar logic establishes that f is a contraction on $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$. These are closed subsets of $\mathbb{R}$, so they form complete metric spaces under the Euclidean metric.

By Banach's fixed point theorem there is exactly one fixed point each on the ranges $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$. So, there are either three fixed points, or two with one of the fixed points being "shared" between two of these intervals, e.g. with $f(\frac{1}{3}) = \frac{1}{3}$.

(vii) Investors with £200000 of assets are able to acquire visas (under some other conditions) to migrate to the United Kingdom. People residing outside the UK receive labour income w each period, and choose how much to consume c and save a', and whether to migrate to the UK. Their utility each period is $u(c)$, which is discounted at rate β. Let $M(a)$ be the value of living in the UK as a migrant with assets a. Both u and M are bounded and concave. The value of assets to a foreigner $V(a)$ is characterised by the Bellman equation

$$V(a) = \begin{cases} M(a) & \text{if } a \geq 200000 \\ \max_{a'} u(a + w - a') + \beta V(a') & \text{if } a \in [0, 200000). \end{cases}$$

You hope to prove that V is concave with the following strategy – which turns out not to work:

(a) Prove that the Bellman operator is a contraction in $(B(\mathbb{R}_+), d_\infty)$.

(b) Prove that (X, d_∞) is a complete metric space, where

$$X = \{V \in B(\mathbb{R}_+) : V \text{ is concave and } V(a) = M(a) \text{ for } a \geq 200000\}.$$

(c) Prove that the Bellman operator is a self-map on X.

(d) Apply Banach's fixed point theorem.

Which step(s) succeed and which step(s) fail? *You will get credit for checking as many steps as you can.*

Comment. For the first part, most students completely ignored $M(a)$, and just replicated the proof of Blackwell's lemma.

For the second part, a common mistake was to think about the convergence of asset holdings of time. This is an interesting thing to study, but is not about the question at all. Convergence of value functions is about repeatedly refining an initial guess.

Few people attempted the third part.

Answer.

(a) This step fails, although it can be fixed with minor amendments.

Let $\Gamma(V)$ be the Bellman operator. If the value function V is poorly behaved (by being discontinuous), there might not be an optimal choice at a, and hence $\Gamma(F)(a)$ is not well-defined.

On the other hand, if we amend the Bellman equation by replacing "max" with "sup", then Blackwell's Lemma can be adapted to make this step work.

(b) This step works. It suffices to prove that X is a closed set. Suppose $V_n \in X$ and $V_n \to V^*$. We must show that $V^* \in X$, i.e. that V^* is concave and $V(a) = M(a)$ for $a \geq 200000$. Consider $a_0, a_1 \geq 0$ and $t \in [0,1]$. Since each V_n is concave, we know that

$$tV_n(a_0) + (1-t)V_n(a_1) \leq V_n(ta_0 + (1-t)a_1).$$

Taking limits, we find that

$$tV^*(a_0) + (1-t)V^*(a_1) \leq V^*(ta_0 + (1-t)a_1).$$

Next, pick any $a \geq 200000$. Since each $V_n \in V$, we know that $V_n(a) = M(a)$. It follows that $V^*(a) = \lim_{n\to\infty} V_n(a) = M(a)$.

(c) This step fails, i.e. $\Gamma(X) \not\subseteq X$. In particular, if $M'(200000)$ is too steep – e.g. bigger than $\frac{M(2000000)-u(w)}{200000} \leq V(0)$ – then there is no way to fit a concave function V through these points.

(d) This step would succeed if the other steps all worked. Specifically, suppose that (i) X were a closed subset of a complete metric space (and hence complete itself), and (ii) $\Gamma : X \to X$ were a self-map. These imply that Γ is a contraction on X, and by Banach's fixed point theorem would have a unique fixed point $V^* \in X$. This is the same fixed point as before, but we would then know that V^* is concave.

(viii) Prove that the following optimization problem (relating to moral hazard) has an optimal solution:

$$\begin{aligned} &\min_{a,b \in \mathbb{R}_+} p\exp(a) + (1-p)\exp(b) \\ &\text{s.t. } pa + (1-p)b \geq 1, \text{ and} \\ &a - b \geq q, \end{aligned}$$

where $p \in (0,1)$ and $q > 0$.

Comment. Nobody answered this question correctly. The key trick that everybody missed is to think of the domain of (a,b) not as $\mathbb{R}^2_+$, but rather as the set of (a,b) that satisfy the two constraints.

Answer. Let C be the set of points (a,b) satisfying the first constraint, and D the points satisfying the second constraint. Since $C = f^{([1,\infty))}$ is the inverse image of a closed set on a continuous funtion $f(a,b) = pa + (1-p)b$, we conclude that C is closed. Similarly, D is closed. So the set of feasible points, $C \cap D$ is closed.

Note that $(a,b) = (1+q,1) \in C \cap D$. So we can add a slack constraint that the point must be better than $(1+q,1)$. This new constraint ensures that the feasible points set is bounded. Then by the Bolzano-Weierstrass theorem, the feasible point set is compact.

The objective is continuous, so by the Extreme Value Theorem, a solution exists.

Question 28. (Microeconomics 1, December 2017)

Several identical households enjoy ice cream more in summer than winter, and enjoy soup in winter more than summer. Households are endowed with cows and fishing boats. Ice cream is made from cows. Soup is made from fishing boats. There are only two time periods (winter and summer).

(i) Formulate a competitive equilibrium model of cows, boats, ice cream and soup during summer and winter.

Comment. A common mistake was to assume that households have the same preferences in summer and winter ($u_0 = u_1$ in my notation), even though the question explicitly said otherwise.

My sample solution does not specify exactly what u_0 and u_1 are. What matters is that this formulation is general enough to accommodate having different preferences in summer and winter. Note that a utility function of the form $u(i_0, s_0) + \beta u(i_1, s_1)$ is not enough, because it cannot express the idea that icecream is better in summmer and soup is better in winter.

A common choice was to formulate the model using home production. This is fine, but it complicates everything. It's important to pick the simplest version of the model that you can.

Answer. Households. There are n identical households which live in periods $t \in \{0, 1\}$, where 0 is summer and 1 is winter. Each household is endowed with b boats and c cows, which it rents at prices r_t^b and r_t^c. It also receives dividends Π/n, where $\Pi = \pi^i + \pi^k$ (defined below). It spends these resources on icecream i_t and soup s_t at prices p_t^i and p_t^s, which give utility $u_0(i_0, s_0) + u_1(i_1, s_1)$. The households' maximisation problem is

$$\begin{aligned} &\max_{i_0,i_1,s_0,s_1} u_0(i_0, s_0) + u_1(i_1, s_1) \\ &\text{s.t. } i_0 p_0^i + i_1 p_1^i + s_0 p_0^s + s_1 p_1^s \leq (r_0^b + r_1^b)b + (r_0^c + r_1^c)c + \tfrac{\Pi}{n}. \end{aligned}$$

Dairy. A dairy rents C_t cows to produce $I_t = f(C_t)$ units of icecream in time period t. Its profit function is

$$\pi^i(p_0^i, p_1^i, r_0^c, r_1^c) = \max_{C_0,C_1} p_0^i f(C_0) + p_1^i f(C_1) - r_0^c C_0 - r_1^c C_1.$$

Kitchen. A kitchen rents B_t boats to produce $S_t = g(B_t)$ units of soup in time period t. Its profit function is

$$\pi^k(p_0^s, p_1^s, r_0^b, r_1^b) = \max_{B_0,B_1} p_0^s g(B_0) + p_1^s g(B_1) - r_0^b B_0 - r_1^b B_1.$$

Equilibrium. Prices $(p^i_t, p^s_t, r^c_t, r^b_t)_{t\in\{0,1\}}$ and quantities $(i_t, s_t, B_t, C_t)_{t\in\{0,1\}}$ constitute an equilibrium if

$$
\begin{aligned}
ni_0 &= f(C_0)\\
ni_1 &= f(C_1)\\
ns_0 &= g(B_0)\\
ns_1 &= g(B_1)\\
nb &= B_0\\
nb &= B_1\\
nc &= C_0\\
nc &= C_1.
\end{aligned}
$$

(ii) Suppose that the boat, cow, and icecream markets clear. Does this imply that the soup markets clear?

Answer. No.

Comment. For example, there could be excess demand for soup in winter and excess supply in summer.

(iii) Reformulate the households' problem by constructing a value function for both time periods, which are connected via a Bellman equation.

Answer. Consider the value of holding money m in period 1,

$$
\begin{aligned}
V_1(m_1, p^i_1, p^s_1, r^c_1, r^b_1) = &\max_{i_1,s_1} u_1(i_1, s_1)\\
&\text{s.t. } i_1p^i_1 + s_1p^s_1 \le r^b_1 b + r^c_1 c + m_1.
\end{aligned}
$$

Then the indirect utility function in period 0 is

$$
\begin{aligned}
V_0(p^i_0, p^s_0, r^c_0, r^b_0; p^i_1, p^s_1, r^c_1, r^b_1) = &\max_{i_0,s_0,m_1} u_0(i_1, s_1) + V_1(m_1, p^i_1, p^s_1, r^c_1, r^b_1)\\
&\text{s.t. } i_0p^i_0 + s_0p^s_0 + m_1 \le r^b_0 b + r^c_0 c + \tfrac{\Pi}{n}.
\end{aligned}
$$

(iv) How does the winter supply of icecream change when the winter price of soup increases?

Comment. Most students did not read the question properly, and answered as if the question were asking about the supply of soup.

Answer. The winter price of soup does not appear in the dairy's problem, so it has no effect.

(v) Is the Pareto frontier of this economy a convex set (under appropriate convexity assumptions about preferences and production)?

Comment. Most students got this wrong. Perhaps students were confusing the utility possibility set (which is convex, at least with free disposal) with its frontier.

Answer. No, assuming that the flow utility function u is strictly concave. Let $\mathcal{U}$ be the Pareto frontier. Let $\hat{u} = u_0(f(C_0), g(B_0)) + u_1(f(C_1), g(B_1))$, and $\bar{u} = u_0(0,0) + u_1(0,0)$. Allocating all resources to household n would give a utility vector of $U_n = (\hat{u}, \bar{u}, \cdots, \bar{u})$. Since these are efficient, we know that each $U_n \in \mathcal{U}$.

Now, consider $U' = \frac{1}{2}U_1 + \frac{1}{2}U_2$. In this case, households 1 and 2 would receive utility

$$\begin{aligned}
&\tfrac{1}{2}[u_0(0,0) + u_1(0,0)] + \tfrac{1}{2}[u_0(f(C_0), g(B_0)) + u_1(f(C_1), g(B_1))] \\
&< u_0(\tfrac{1}{2}f(C_0), \tfrac{1}{2}g(B_0)) + u_1(\tfrac{1}{2}f(C_1), \tfrac{1}{2}g(B_1))] \\
&= u^*.
\end{aligned}$$

So households 1 and 2 strictly prefer u^* over $\frac{1}{2}\bar{u} + \frac{1}{2}\hat{u}$. This means that $(u^*, u^*, \bar{u}, \cdots, \bar{u})$ is feasible and Pareto dominates U'. So $U' \notin \mathcal{U}$. We conclude that $\mathcal{U}$ is not a convex set.

(vi) Prove that there is only one competitive equilibrium (under appropriate convexity assumptions about preferences and production).

Comment. There are two critical ingredients: (i) equilibria are efficient (the first welfare theorem) and (ii) equilibria are symmetric if households are all the same. Most students missed one of these ingredients out.

Answer. Assume the utility function is strictly concave and the production is strictly increasing.

By the first welfare theorem, all equilibrium are Pareto efficient. Since households are identical, they all acquire the same utility in equilibrium. Since the equilibrium allocation is efficient and gives equal utility to all households, it be a solution to the egalitarian social planner's problem,

$$\max_{(i_{ht},s_{ht})_{h\in\{1,\cdots,n\},t\in\{0,1\}}} \sum_{h=1}^{n}[u_0(i_{h0},s_{h0}+u_1(i_{h1},s_{h1}]$$

$$\text{s.t. } \sum_{h=1}^{n} i_{h0} = f(nc)$$

$$\sum_{h=1}^{n} i_{h1} = f(nc)$$

$$\sum_{h=1}^{n} s_{h0} = g(nb)$$

$$\sum_{h=1}^{n} s_{h1} = g(nb).$$

Since the constraint is linear and the objective is strictly concave, it has a unique solution.

(vii) * Consider any metric space (X,d), and any two sets A and B with $A \subseteq B \subseteq X$. Prove that if A is open in (X,d), then A is open in (B,d).

Comment. Most students did not attempt this question.

Answer. Pick any $a \in A$. We must find some $r > 0$ such that

$$\{b \in B : d(a,b) < r\} \subseteq A.$$

Since A is open in (X,d), there is some $s > 0$ such that

$$\{x \in X : d(a,x) < s\} \subseteq A.$$

Pick $r = s$. Since $B \subseteq X$, it follows that

$$\{x \in B : d(a,x) < r\} \subseteq \{x \in X : d(a,x) < r\} \subseteq A$$

as required.

(viii) * Let $f : X \to X$ be a function on a complete metric space (X,d). Suppose that $g(x) = f(f(x))$ is a contraction. Prove that f has a unique fixed point.

Comment. Most students did not attempt this question.

Answer. By Banach's fixed point theorem, g has a unique fixed point, x^*. So $f(f(x^*)) = x^*$. Since fixed points of f are also fixed points of g, we conclude that x^* is the only possible fixed point of f.

It remains to show that x^* is a fixed point of f. Now, consider $y = f(x^*)$. We need to prove that $y = x^*$. Notice that

$$g(y) = g(f(x^*)) = f(f(f(x^*))) = f(g(x^*)) = f(x^*) = y.$$

So y is also a fixed point of g. But g has exactly one fixed point, namely x^*. So we conclude that $y = x^*$, as required.