

Reverse Calculus and Nested Optimization^{*}

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Abstract

Nested optimization problems arise when an agent must take into account the effect of their decisions on their own future behaviour, or the behaviour of others. In these problems, calculating marginal costs and benefits involves differentiating the solutions to nested problems. But are these solutions differentiable functions? We develop a tool called *Reverse Calculus*, and establish first-order conditions for (i) a Stackelberg leader considering the follower's best response function, (ii) a sovereign borrower considering its own future default policy, and (iii) non-convex dynamic programming problems.

Keywords: first-order conditions, non-convex dynamic programming, Stackelberg problems, unsecured credit

JEL: C61, F34, L13

1. Introduction

A fundamental insight of economics is that optimal choices occur where marginal benefit equals marginal cost. In simple economies, both sides of this first-order condition are exogenous, and can be assumed to exist. In recursive macroeconomies, the marginal benefit of preparing for the future is endogenous, and envelope theorems have established its existence in well-behaved convex settings. But there are many other important economic problems in which policy or value functions appear in optimization objectives or constraints. In these nested optimization problems, it is unknown whether first-order conditions are applicable.

Threats to first-order conditions. *Jumps* can arise in objective functions even if all of the model primitives are continuous. For example, consider the Stackelberg duopoly game,

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which we study in detail in [Section 3.1](#). The follower’s policy function appears inside the leader’s objective function. Several conditions (such as strictly convex costs) are required to ensure that the follower’s policy function is continuous. Otherwise, the follower’s policy function is discontinuous, so the leader’s objective function is not continuous, let alone differentiable.

Kinks can arise in objective functions even if all model primitives are differentiable. This occurs when a continuous choice is taken alongside a discrete choice. For example, consider a Stackelberg leader who can build his factory in China or Europe. Assume each location has a differentiable cost function. Despite this assumption, the firm’s overall cost function has a kink at the quantity where both locations are equally costly.

Hidden jumps and kinks arise when the objective is differentiable, but ingredients such as benefit and cost are not. For example, suppose that the Stackelberg leader does not know the follower’s cost curve. He assigns probabilities to two different follower cost curves, and hence to two different leader benefit functions. Even if the expected benefit of selling output is differentiable, this does *not* imply that the ex post benefit curves are differentiable. They might have kinks or jumps that cancel each other out. In this case, it is impossible to write a meaningful first-order condition involving marginal costs and marginal benefits.

Boundary problems arise when decision makers prefer to make boundary choices, such as exhausting capacity constraints. Since first-order conditions only apply to interior solutions, economists often steer decision makers to the interior by imposing Inada conditions. This is problematic as the relevant envelope theorems depend on placing uniform bounds on derivatives. In other words, economists wishing to apply first-order conditions have an uncomfortable choice between (i) assuming Inada conditions hold to ensure that all solutions are interior, or (ii) assuming that Inada conditions do not hold, to ensure that derivatives exist.

These threats are common place in important economic problems. For example, all four threats arise in the unsecured credit market model of [Arellano \(2008\)](#), which we study in [Section 3.2](#). First, the borrower’s future default policy appears in his objective, because it determines default risk and hence interest rates. There is no a priori reason why his policy would be differentiable. Second, the borrower has a discrete choice – whether to honour or default on debts owed – leading to kinks in his value function. Third, even if the objective is differentiable, the default policy and value function might have jumps or kinks that cancel each other out. Fourth, Arellano focuses on a utility function that satisfies the Inada conditions. These four features of Arellano’s model pose difficulties to applying existing tools.

Techniques. We devise a recipe that addresses these threats to first-order conditions. The recipe makes use of two new techniques. Our **Differentiable Sandwich Lemma** reformulates a classical result about subderivatives in simple form that is useful for establishing first-order conditions.¹ It establishes that a function F is differentiable at a point \bar{c} if it is

¹ Specifically, [Rockafellar and Wets \(1998, Proposition 8.5\)](#) establish that a function is subdifferentiable if and only if it has a differentiable lower support function. (See [Clausen and Strub \(2016, Appendix F\)](#) for

sandwiched between two differentiable functions U and L , as depicted in Figure 1a. Specifically, the lemma applies if the two functions, which we call differentiable upper and lower support functions, satisfy (i) $U(\bar{c}) = F(\bar{c}) = L(\bar{c})$, (ii) $U(c) \geq F(c) \geq L(c)$ for all c , and (iii) L and U are differentiable at \bar{c} . Our lemma accommodates functions that are neither continuous nor differentiable in any open neighbourhood of \bar{c} , as depicted in Figure 1b.

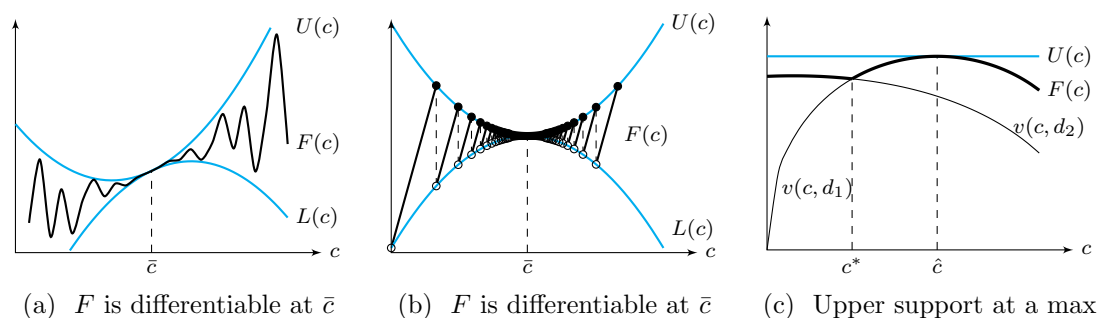


Figure 1: Differentiable Sandwich Lemma

The Differentiable Sandwich Lemma has a natural economic meaning. Suppose a decision maker wants to choose c to maximise $F(c)$. The lemma establishes that at any \bar{c} , there are only three possibilities:²

1. F is differentiable at \bar{c} .
2. F lacks a differentiable upper support function at \bar{c} , as depicted in Figure 1c at c^* . In particular, $U(c) = F(\bar{c})$ is not an upper support function. So \bar{c} is not an optimal choice, and we call \bar{c} a **bad kink**.
3. F lacks a differentiable lower support function at \bar{c} . In this case, any differentiable function that correctly values \bar{c} must overvalue some choices near \bar{c} . Therefore, this type of kink makes deviating from \bar{c} less attractive. So \bar{c} might be a good choice, and we call \bar{c} a **good kink**.

If F is not an objective function, then a similar interpretation is possible, where good or bad depends on whether large quantities are good or bad for the decision maker.

The lemma is well suited to studying optimal choices. Suppose that the decision maker must make a continuous choice $c \in \mathbb{R}$ followed by a discrete choice $d \in D$. Assume that his utility function $v(c, d)$ is differentiable in c for each discrete choice d . Let $F(c) = \max_{d \in D} v(c, d)$ be the value after choosing c . Notice that at an optimal choice (\hat{c}, \hat{d}) , the value function is sandwiched between the horizontal line $U(c) = F(\hat{c})$ and $v(\cdot, \hat{d})$. This sandwich is depicted in Figure 1c, where $\hat{d} = d_2$. The differentiable upper support function rules out \hat{c} being a bad kink – otherwise, there would be a better choice near \hat{c} . The

a simpler proof.) It is then straightforward to show that a function is differentiable if and only if it is both sub- and superdifferentiable; see Kruger (2003).

² Note that \bar{c} can be both a good and a bad kink, e.g. $F(c) = c \sin \frac{1}{c}$ at $\bar{c} = 0$.

differentiable lower support function rules out \hat{c} being a good kink, because the optimal discrete choice \hat{d} is available near \hat{c} . Thus, the Differentiable Sandwich Lemma establishes that F is differentiable at \hat{c} . [Milgrom and Segal \(2002, Corollary 2\)](#) previously drew this conclusion for the special case that $\{v(\cdot, d)\}_{d \in D}$ is equi-differentiable and has uniformly bounded derivatives. Their redundant conditions conflict with Inada conditions. In contrast, our Differentiable Sandwich Lemma is applicable to problems with discrete choices and Inada conditions.³

Our second and most novel innovation, **Reverse Calculus**, is the opposite of normal calculus. Whereas normal calculus establishes that $H(c) = F(c) + G(c)$ is differentiable if F and G are differentiable, Reverse Calculus establishes that F and G are differentiable if H is differentiable. The main requirement for Reverse Calculus is that each ingredient function must have an appropriate differentiable support function. For example, if $H(c) = F(c) + G(c)$, then we require F and G have differentiable lower support functions f and g at \bar{c} , depicted in [Figure 2](#). Under these conditions, F is sandwiched between f and $H - g$, and is therefore differentiable at \bar{c} . Similarly, G is differentiable at \bar{c} . We develop a Reverse Calculus for many standard operations, including addition, multiplication, function composition and upper envelopes.

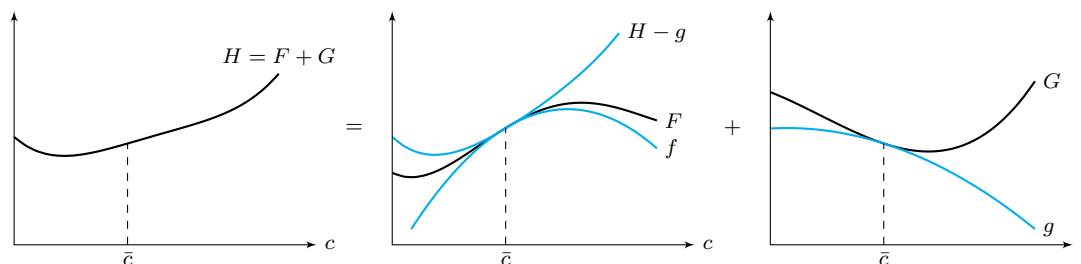


Figure 2: Reverse calculus: F is differentiable at \bar{c} .

Reverse calculus is well suited to studying nested optimization problems in which the objective involves the solutions to other optimization problems. In these problems, it is insufficient to establish that the objective function (e.g. H) is differentiable. Meaningful first-order conditions require us to establish that all of the nested solution functions (e.g. F and G) are differentiable.

We combine these two tools into a **recipe for deriving first-order conditions in nested optimization problems**. Previous tools either focused on establishing the differentiability of value functions (as in the envelope theorems of [Mirman and Zilcha \(1975\)](#); [Benveniste and Scheinkman \(1979\)](#); [Milgrom and Segal \(2002\)](#)), or of policy functions in smooth concave problems ([Araújo and Scheinkman \(1977\)](#); [Santos \(1991\)](#)). The first half of the recipe applies the Differentiable Sandwich Lemma to establish that the objective is differentiable at optimal choices. The second half of the recipe applies Reverse Calculus

³ Our result accommodating Inada conditions has been applied in computational research. [Fella \(2014\)](#) uses our [Theorem 6](#) to construct a value function interpolation algorithm. Fella’s algorithm has been adapted to higher dimensions by [Drue Dahl and Jørgensen \(2017\)](#).

to establish that the nested solution functions are differentiable at optimal choices. The main ingredient required for using the recipe is the collection of differentiable upper or lower support functions, one for each nested solution function.

How are these support functions to be found? One approach is based on “lazy decision maker” constructions that involve unreactive policy functions. [Benveniste and Scheinkman’s \(1979\)](#) applied this strategy to construct a differentiable lower support function for value functions. This is a special case of a more general approach of constructing optimistic or pessimistic valuations. For example, in our analysis of unsecured credit markets, we construct support functions for optimal default rules based on a pessimistic value of honouring debts. Similarly, we construct a support function for a Stackelberg follower policy based on pessimistic beliefs about when the follower is capacity constrained.

Outline. [Section 2](#) formally specifies and proves the lemmas used in the recipe. [Section 3.1](#) describes our recipe for deriving first-order conditions using a Stackelberg duopoly as a running example. [Section 3.2](#) applies the recipe to Arellano’s model of unsecured credit markets. [Section 3.3](#) provides an elementary proof of the [Benveniste and Scheinkman \(1979\)](#) envelope theorem for convex dynamic programming problems. [Section 3.4](#) uses our recipe to establish first-order conditions in non-convex stochastic dynamic programming problems. [Section 4](#) concludes. [Appendix A](#) presents Reverse Calculus rules for convex combinations and function composition.

2. Techniques

This section presents the techniques used in our recipe, namely the Differentiable Sandwich Lemma and Reverse Calculus.

2.1. Differentiable Sandwich Lemma

The Differentiable Sandwich Lemma establishes that if a function $F : C \rightarrow \mathbb{R}$ is sandwiched at some point \bar{c} between two differentiable functions, then F is differentiable at \bar{c} . The domain is typically a subset of \mathbb{R}^n , although we can study the (Fréchet) differentiability of functions on any Banach space.

Lemma 1 (Differentiable Sandwich Lemma). *Let $N \subseteq C$ be an open neighbourhood of \bar{c} . If*

1. $U(\bar{c}) = F(\bar{c}) = L(\bar{c})$,
2. $U(c) \geq F(c) \geq L(c)$ for all $c \in N$, and
3. L and U are differentiable at \bar{c} ,

then F is differentiable at \bar{c} with $F'(\bar{c}) = L'(\bar{c}) = U'(\bar{c})$.

Proof. The difference function $d(c) = U(c) - L(c)$ is locally minimized at \bar{c} . Therefore, $d'(\bar{c}) = 0$ and we conclude $L'(\bar{c}) = U'(\bar{c})$.

Let $m = L'(\bar{c}) = U'(\bar{c})$. For all Δc with $\bar{c} + \Delta c \in N$,

$$\frac{L(\bar{c} + \Delta c) - F(\bar{c}) - m \Delta c}{\|\Delta c\|} \leq \frac{F(\bar{c} + \Delta c) - F(\bar{c}) - m \Delta c}{\|\Delta c\|} \leq \frac{U(\bar{c} + \Delta c) - F(\bar{c}) - m \Delta c}{\|\Delta c\|}. \quad (1)$$

Consider the limits as $\Delta c \rightarrow 0$. Since $L'(\bar{c}) = U'(\bar{c}) = m$, the limits of the first and last fractions are 0. By Gauss' Squeeze Theorem, we conclude that the limit in the middle is also 0, and hence that F is differentiable at \bar{c} with $F'(\bar{c}) = m$. \square

2.2. Reverse Calculus

Calculus involves rules such as “if F and G are differentiable at \bar{c} , then $H(c) = F(c) + G(c)$ is also differentiable at \bar{c} .” Reverse calculus rules go in the opposite direction. We provide the most important rules here, and additional rules for convex combinations and more general function composition in [Appendix A](#). We omit the corresponding rules for subtraction and division, which involve both upper and lower support functions.

Lemma 2 (Reverse Calculus). *Suppose $F : C \rightarrow \mathbb{R}$, and $G : C \rightarrow \mathbb{R}$ have differentiable lower support functions f , and g respectively at \bar{c} .*

1. *If $H(c) = F(c) + G(c)$ is differentiable at \bar{c} , then F is differentiable at \bar{c} .*
2. *If $H(c) = F(c)G(c)$ is differentiable at \bar{c} and $F(\bar{c}) > 0$ and $G(\bar{c}) > 0$, then F is differentiable at \bar{c} .*
3. *If $H(c) = \max\{F(c), G(c)\}$ is differentiable at \bar{c} and $F(\bar{c}) = H(\bar{c})$, then F is differentiable at \bar{c} .*
4. *If $H(c) = J(F(c))$ is differentiable at \bar{c} and $J : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable at $F(\bar{c})$ with $J'(F(\bar{c})) \neq 0$, then F is differentiable at \bar{c} .*

Proof. Let f and g be differentiable lower support functions of F and G at \bar{c} . For (i)–(iii), we sandwich F between f and an appropriate differentiable upper support function U and apply the Differentiable Sandwich Lemma ([Lemma 1](#)). Appropriate upper support functions are (i) $U(c) = H(c) - g(c)$, (ii) $U(c) = H(c)/g(c)$, and (iii) $U(c) = H(c)$.

For (iv), $F(c) = J^{-1}(H(c))$ is differentiable at \bar{c} by the inverse function theorem and the chain rule. \square

3. Applications

3.1. Stackelberg Duopoly

This section illustrates our recipe for deriving first-order conditions in two Stackelberg duopoly models. The models are simple, so the reader can focus on understanding the recipe. The first version of the model focuses on explaining the role of the Differentiable Sandwich Lemma. The second version focuses on Reverse Calculus.

3.1.1. Stackelberg with a Capacity Constraint

In Stackelberg duopoly games, the leader's first-order condition involves the derivative of the follower's policy function. But this first-order condition is only valid if the follower's policy is differentiable (at the leader's optimal quantity). Other related problems are explored by [Kydlan and Prescott \(1977\)](#) and [Ljungqvist and Sargent \(2012, Chapter 19\)](#).

We study a Stackelberg duopoly in which the follower is capacity constrained. The leader chooses her output y_1 first. Then the follower chooses his output y_2 , subject to an exogenous capacity constraint $y_2 \leq Y$. The leader and follower pay production costs of $C(y_1, a)$ and $C(y_2, b)$ respectively; we use $z \in \{a, b\}$ to index the production functions. The output is sold at the market price, $P(y_1 + y_2)$. Firm i earns a profit of $\pi(y_1, y_2, i, z) = y_i P(y_1 + y_2) - C(y_i, z)$. The follower chooses $y_2 = f(y_1)$ by solving

$$f(y_1) = \arg \max_{y_2} \pi(y_1, y_2, 2, b) \quad (2)$$

and the leader chooses y_1 to maximise

$$\phi(y_1) = \pi(y_1, f(y_1), 1, a) = y_1 P(y_1 + f(y_1)) - C(y_1, a). \quad (3)$$

Can we write down necessary first-order conditions for these choices? If so, the follower's and leader's first-order conditions would be

$$P(y_1 + \hat{y}_2) + P'(y_1 + \hat{y}_2)\hat{y}_2 = C_{y_2}(\hat{y}_2, b) \quad \text{and} \quad (4)$$

$$P(\hat{y}_1 + f(\hat{y}_1)) + P'(\hat{y}_1 + f(\hat{y}_1))(1 + f'(\hat{y}_1))\hat{y}_1 = C_{y_1}(\hat{y}_1, a). \quad (5)$$

But does the derivative $f'(\hat{y}_1)$ of the follower's policy exist? We can not simply assume the follower chooses a differentiable policy. In fact, the policy has a kink where he would voluntarily choose the maximum capacity, as depicted in [Figure 3a](#). At this kink, the follower policy is less aggressive than any differentiable approximation (that agrees at the kink). This translate into a bad kink in the leader's objective, depicted in [Figure 3b](#). We will show that the leader avoids this bad kink, because there are better choices nearby. We conclude that the leader chooses a differentiable point, and first-order conditions hold.

The only novelty in this example is the capacity constraint. Suppose we had imposed a minimum quantity instead. Like before, the follower's policy would have a kink where he voluntarily chooses the minimum quantity. However, the constraint would now make the follower react more aggressively to the leader's output choice. The kink might be a good choice for the leader, because the nearby choices provoke a non-differentiably aggressive reaction. In this case, first-order conditions do not hold at the leader's optimal choice.

Theorem 1. *Assume the demand function P and cost functions $C(\cdot, z)$ are twice differentiable, and strictly concave and convex respectively. If $(\hat{y}_1, \hat{y}_2(b))$ are equilibrium quantities, then (i) ϕ and f are differentiable at \hat{y}_1 , and (ii) \hat{y}_1 satisfies (5).*

Proof. The proof is a simple example of the following recipe used in all of the applications.

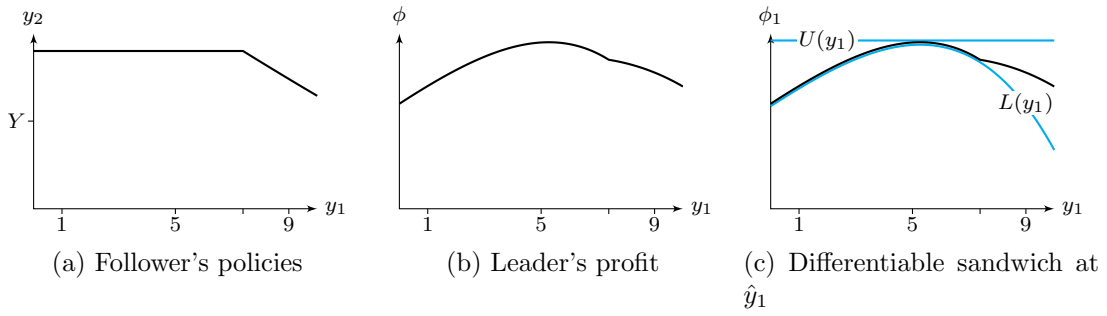


Figure 3: A Stackelberg duopoly with a capacity constraint

Ingredients: Differentiable support functions. We construct differential support functions for the leader’s objective and the follower’s policy, to rule out \hat{y}_1 being a bad kink for the leader.

First, we construct a differential upper support function F for f at \hat{y}_1 . From the leader’s point of view, f is better than the differentiable policy F because it involves less aggressive competition. Thus, we will rule out \hat{y}_1 being a bad kink of f . Now, f is the lower envelope of the constant policy Y and the unconstrained best response policy, i.e. it selects the less aggressive of the two policies at \hat{y}_1 . Since the less aggressive policy is more aggressive than f , it is an upper support function at \hat{y}_1 . Moreover, it is differentiable because both policies are differentiable (under the assumptions above). We conclude that the less aggressive policy is a differentiable upper support function for f at \hat{y}_1 .

Second, we construct a differential lower support function L for the leader’s objective ϕ at \hat{y}_1 . Since F is more aggressive than f , it gives lower profits to the leader. Therefore $L(y_1) = y_1 P(y_1 + F(y_1)) - C(y_1, a)$ is a differentiable lower support function for ϕ .

Step 1: Upper support function. The constant function $U(y_1) = \phi(\hat{y}_1)$ is a differentiable upper support function for ϕ at \hat{y}_1 . This rules out \hat{y}_1 being a good kink of ϕ – otherwise there would be a better choice than \hat{y}_1 nearby.

Step 2: Apply the Differentiable Sandwich Lemma. The support functions L and U form a sandwich around ϕ , as illustrated in Figure 3c. By the Differentiable Sandwich Lemma, the leader’s objective is differentiable at \hat{y}_1 with $\phi'(\hat{y}_1) = U'(\hat{y}_1)$. Since $U'(\hat{y}_1) = 0$, we deduce the first-order condition $\phi'(\hat{y}_1) = 0$.

Step 3: Apply Reverse Calculus. However, we have not yet established (5), which is a more useful first-order condition. In particular, we have not yet determined whether the nested solution function f is differentiable at \hat{y}_1 . This is a Reverse Calculus problem: we have established that the left side of (3) is differentiable, and we would now like to infer that f on the right side is differentiable. Since there is only one nested solution function, kink cancellation is impossible, so this step particularly easy: we apply the implicit function theorem to (3). This establishes that $y_2 = f(y_1)$ is a differentiable function of y_1 at \hat{y}_1 . \square

3.1.2. Stochastic Stackelberg

In the Stackelberg model above, there was only one nested solution function, namely the follower's policy. This made it easy to rule out kink cancellation. We now assume that the follower might have high or low costs, so the leader must anticipate two nested solution functions. Ruling out kink cancellation will require careful use of Reverse Calculus.

Suppose that the leader knows that the follower's cost function is either $C(\cdot, b)$ or $C(\cdot, c)$, with probability p_b and p_c respectively. The follower chooses $y_2 = f(y_1, z)$ by solving

$$f(y_1, z) = \arg \max_{y_2} \pi(y_1, y_2, 2, z) \quad (6)$$

and the leader chooses y_1 to maximise

$$\phi(y_1) = \sum_{z \in \{b, c\}} p_z \pi(y_1, f(y_1, z), 1, a) = \sum_{z \in \{b, c\}} p_z y_1 P(y_1 + f(y_1, z)) - C(y_1, a). \quad (7)$$

We would like to verify that the necessary first-order condition holds:

$$\sum_{z \in \{b, c\}} p_z [P(\hat{y}_1 + f(\hat{y}_1, z)) + P'(\hat{y}_1 + f(\hat{y}_1, z))(1 + f_{y_1}(\hat{y}_1, z))\hat{y}_1] = C_{y_1}(\hat{y}_1, a). \quad (8)$$

Like before, we must establish that the derivatives $f_{y_1}(\hat{y}_1, z)$ of the follower's policies exist.

Theorem 2. *Assume the demand function P and cost functions $C(\cdot, z)$ are twice differentiable, and strictly concave and convex respectively. If $(\hat{y}_1, \hat{y}_2(b), \hat{y}_2(c))$ are equilibrium quantities, then (i) ϕ , $f(\cdot, b)$ and $f(\cdot, c)$ are differentiable at \hat{y}_1 , and (ii) \hat{y}_1 satisfies (5).*

Proof. Ingredients: Differentiable support functions. A differentiable upper support functions $F(\cdot, c)$ at \hat{y}_1 for the new follower policy $f(\cdot, c)$ can be constructed the same was as for $f(\cdot, b)$. A differential lower support function L for the leader's objective ϕ at \hat{y}_1 is $L(y_1) = \sum_{z \in \{b, c\}} p_z y_1 P(y_1 + F(y_1, z)) - C(y_1, a)$.

Step 1: Upper support function. As before, the constant function $U(y_1) = \phi(\hat{y}_1)$ is a differentiable upper support function for ϕ at \hat{y}_1 .

Step 2: Apply the Differentiable Sandwich Lemma. As before, the support functions L and U form a sandwich around ϕ . By the Differentiable Sandwich Lemma, the leader's objective is differentiable at \hat{y}_1 with $\phi'(\hat{y}_1) = U'(\hat{y}_1) = 0$.

Step 3: Apply Reverse Calculus. We need to prove that the two nested solution functions, $f(\cdot, b)$ and $f(\cdot, c)$ are differentiable at \hat{y}_1 . In step 2, we established that both sides of (7) are differentiable. First, we apply the reverse summation rule from Lemma 2. The right side of (7) is the sum of three terms, each of which has a differentiable lower support function. (The first of these functions is $y_1 \mapsto p_b y_1 P(y_1 + F(y_1, b))$.) The rule then implies that each term is differentiable at \hat{y}_1 .

We apply Reverse Calculus rules from Lemma 2 as follows:

- By the reverse product rule, $P(y_1 + f(y_1, b))$ is differentiable at \hat{y}_1 .
- By the reverse chain rule, $y_1 + f(y_1, b)$ is differentiable at \hat{y}_1 .
- By the reverse summation rule, $f(\cdot, b)$ is differentiable at \hat{y}_1 .

A similar procedure establishes that $f(\cdot, c)$ is differentiable at \hat{y}_1 . □

Note that the leader's uncertainty about the follower's costs turns out to be straightforward in this example – all kinks in the follower's policies carry into the leader's objective, so no kinks are hidden by kink cancellation. But it is not always this trivial. For example, in the unsecured credit application in Section 3.2, the budget constraint contains both good and bad kinks.

3.2. Unsecured Credit

Our second application is about unsecured debt contracts where borrowers may decide either to repay in full or to default. The borrower uses debt to smooth consumption against endowment shocks. We focus on markets without collateral such as sovereign debt. The punishment for default is exclusion from the credit market thereafter. Nevertheless, default occasionally occurs so interest paid by the borrower must compensate for the default risk. For this reason, the interest charged is non-linear and determined by a recursive relationship with the borrower's value function. If the interest rates are low, then the borrower's value of honouring debt contracts is high because rolling over debt is cheap. Conversely, if the borrower's value of repaying is high tomorrow, then the default risk today is low. This recursive relationship determines interest rates as a function of loan sizes.

All four threats to first-order conditions from the introduction are present. First, the borrower's future default policy appears in his objective, because it determines default risk and hence interest rates. There is no a priori reason why his policy would be differentiable. Second, the borrower has a discrete choice – whether to honour or default on debts owed – leading to kinks in his value function. Third, even if the objective is differentiable, the default policy and value function might have jumps or kinks that cancel each other out. Fourth, we impose (problematic) Inada conditions so that the borrower makes interior consumption choices.

We find that both nested solution functions – the value function and the interest rate – are differentiable at optimal debt choices, except when the borrower exhausts his risk-free credit. Hence, we derive a first-order condition involving a marginal interest rate and a marginal continuation value. We use this condition to prove that the borrower never exhausts his (risky) credit limit.

We sketch the logic behind these results. First, consider the risk-free limit, i.e. the largest loan which is repaid with certainty. Imagine the borrower decreasing his loan to this limit. We prove that the default probability abruptly stops decreasing at zero. Hence, the interest rate abruptly stops decreasing at the risk-free rate. This is a good kink, and the borrower might choose it.

All remaining kinks are bad, so the borrower would never choose such a kink – there is a better choice nearby. First, consider the value function. It has kinks at states where the borrower is indifferent between honouring or defaulting on debt. But these kinks are bad, because at nearby states, the borrower chooses the better of honouring or defaulting.

But what about the risky part of the interest rate function? The default policy can be described in terms of the default/honour frontier. This frontier is depicted in [Figure 4a](#), where debt is on the horizontal axis, and the endowment shock on the vertical axis. We just argued that the value of honouring debt only contains bad kinks, i.e. kinks where any differentiable function undervalues honouring promises. These kinks lead to bad kinks in the honour frontier, i.e. kinks where any differentiable function misclassifies some nearby honour states as default states. Thus, the interest rate function only has bad kinks. Our main conclusions then follow.

We build on the unsecured credit analysis by [Arellano \(2008\)](#) which is in the tradition of [Eaton and Gersovitz \(1981\)](#). Arellano carefully analyses it theoretically and numerically. She also sketches a Laffer curve for the debt choice, but – without first-order conditions – does not characterize borrower behaviour along it. One paper by [Aguiar *et al.* \(2019\)](#) derives first-order conditions for a simple example, where it is possible to avoid differentiating any policy functions. The following three papers apply some Euler equations, with the first explicitly acknowledging that they lack justification for differentiating the interest rates with respect to loan size. We provide a justification. [Aguiar and Gopinath \(2006\)](#) dropped a detailed discussion of their heuristic Euler equation from their NBER working paper version. Similarly, [Arellano and Ramanarayanan \(2012\)](#) use heuristic Euler equations to compare maturity structures of loans. Finally, [Hatchondo and Martinez \(2009\)](#) discuss an Euler equation, implicitly assuming differentiability of interest rates.

Model. A risk-averse borrower has a differentiable utility function u and discount factor $\beta \in (0, 1)$. The borrower’s marginal value of consumption at zero is infinite, i.e. $\lim_{c \rightarrow 0^+} u_1(c) = \infty$. Every period, the borrower receives an endowment x which is independently and identically distributed with density $f(\cdot)$ on the support $[x^{\min}, x^{\max}]$. We assume the borrower’s endowment is bounded away from zero, i.e. $x^{\min} > 0$. To smooth out endowment shocks, the borrower may take out loans from a lender with deep pockets. We focus our attention on debt contracts of the following form. The borrower promises to pay a lender b' in the following period, although both understand that the borrower only has an incentive to honour the promise if tomorrow’s x' lies in some set H' . Thus, a debt contract consists of (b', H') . The lender is risk-neutral, discounts time at the same rate, and is therefore willing to pay $\beta \int_{H'} f(x') dx' b'$ in return for the promise. If the borrower defaults – regardless of whether $x' \in H'$ – he is excluded from credit markets thereafter. We also accommodate an additional exogenous sanction of $s \geq 0$ units of consumption every period for defaulting, which reflects the difficulty of settling non-financial transactions without credit.⁴ The borrower’s autarky

⁴ Exogenous sanctions are often included in unsecured credit models, so we include them to show the generality of our technique. Without them, [Bulow and Rogoff \(1989\)](#) show that exclusion from credit markets alone is an insufficient punishment for enforcing debt contracts if the borrower can make private investments.

value after defaulting is

$$A(x) = u(x - s) + \beta \int A(x')f(x')dx'. \quad (9)$$

The lender only agrees to the contract (b', H') if the borrower has an incentive to honour the promise for the proposed endowments H' . Specifically, the borrower's value of repaying b' at an honour endowment $x' \in H'$, denoted $V(b', x')$, should not be less than the autarky value $A(x)$. The borrower's value of honouring debts is therefore⁵

$$\begin{aligned} V(b, x) = & \sup_{c, b', H'} u(c) + \beta \int \max \{A(x'), V(b', x')\} f(x')dx', \\ \text{s.t. } & c + b = x + \left[\beta \int_{H'} f(x') dx' \right] b', \\ & V(b', x') \geq A(x') \text{ for all } x' \in H', \\ & b' \leq b^{\text{ponzi}}. \end{aligned} \quad (10)$$

The last constraint rules out Ponzi schemes and the b^{ponzi} parameter may be arbitrarily large.

Reformulation. We reformulate this problem by making two simplifications. First, [Arelano \(2008, Proposition 3\)](#) established that because x is IID, the honour set H' chosen by the borrower is determined by a cut-off rule $y(\cdot)$ so that the borrower honours his debt at state (b', x') if and only if $x' \geq y(b')$. In other words, the borrower only ever chooses debt contracts of the form $(b', H') = (b', [y(b'), x^{\text{max}}])$, so debt contracts are characterized by b' alone. This means we may denote the price of debt $q(b')$ as a function of b' . Second, we substitute the budget constraint into the objective, so that the borrower's only choice is his future debt obligation b' . The reformulated problem becomes

$$V(b, x) = \sup_{b' \leq b^{\text{ponzi}}} \phi(b'; b, x), \quad (11)$$

where

$$\phi(b'; b, x) = u(x - b + q(b')b') + \beta W(b'), \quad (12a)$$

$$W(b') = \int \max \{A(x'), V(b', x')\} f(x')dx', \quad (12b)$$

$$q(b') = \beta[1 - F(y(b'))], \quad (12c)$$

$$y(b') = \min (\{x' \in [x^{\text{min}}, x^{\text{max}}] : V(b', x') \geq A(x')\} \cup \{x^{\text{max}}\}). \quad (12d)$$

⁵ We mention some technicalities: (i) the borrower should be constrained to choosing a measurable honour set, and (ii) since we focus on first-order conditions, we take it for granted that the value function in the sequence problem is the unique solution to the Bellman equation.

We denote optimal policy functions by $\hat{b}'(b, x)$.⁶

The objective (12a) has two nested solution functions, q and W , which we will show are not globally differentiable. The value function W has downward kinks at states of indifference between honouring and defaulting, as depicted in Figure 1c. Similarly, we have no a priori knowledge of the differentiability of the debt price q .

We will follow the four steps of the recipe to establish that at optimal choices, first-order conditions hold and that $\phi(\cdot, b, x)$, q and W are differentiable. However, we find that there is one exception: the debt price exhibits an upward kink at the risk-free credit limit. This means that first-order conditions are inapplicable when the borrower chooses to exhaust his risk-free credit limit.

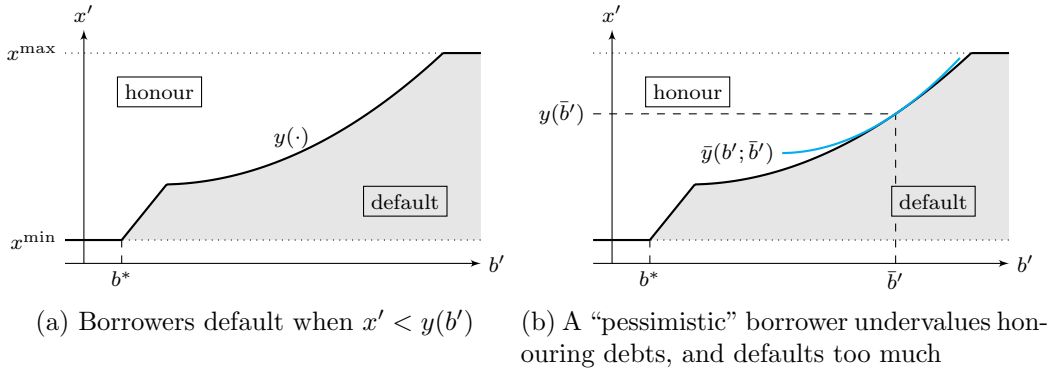


Figure 4: The default cut-off rule

Ingredients: Differentiable Lower Support Functions. The most important ingredient of the recipe is providing appropriate differentiable support functions for the nested solution functions to rule out good kinks. The borrower prefers to sell his promises at high prices which must reflect low default probabilities, and he prefers high continuation values. Therefore, ruling out good kinks involves finding differentiable lower support functions for the debt price $q(\cdot)$ and the continuation value $W(\cdot)$, and a differentiable upper support function for the default cut-off rule $y(\cdot)$.

For debts below some threshold b^* , the borrower always honours his obligations, so the cut-off $y(\cdot)$ is constant and hence differentiable on $(-\infty, b^*)$. At each debt level $\bar{b}' > b^*$, we now construct a differentiable upper support function for $y(\cdot)$. We consider a pessimistic borrower that undervalues honouring debts, and hence uses a higher cut-off than $y(\cdot)$. Specifically we consider a pessimistic borrower who incorrectly anticipates the state to be $(b', x') = (\bar{b}', y(\bar{b}'))$, i.e. he anticipates his state will be on the cut-off. In unanticipated states, he chooses his debt to be $\hat{b}''(\bar{b}', y(\bar{b}'))$ independently of the realized endowment x' . His consumption is adjusted by the differences from the anticipated endowment and debt. This pessimistic borrower's value function is

$$L(b', x'; \bar{b}') = u(x' - b' + q(\bar{b}'')\bar{b}'') + \beta W(\bar{b}''). \quad (13)$$

⁶ The borrower might be indifferent between several optimal policies.

Since the pessimistic borrower undervalues honouring debts, his honour cut-off $\bar{y}(\cdot; \bar{b}')$ implicitly defined by

$$L(b', \bar{y}(b'; \bar{b}'); \bar{b}') = A(\bar{y}(b'; \bar{b}')) \text{ for all } b' \quad (14)$$

provides an upper support function for the cut-off $y(\cdot)$ at \bar{b}' that involves defaulting too often, depicted in [Figure 4b](#). Since the pessimistic borrower's value function is differentiable, the implicit function theorem implies that $\bar{y}(\cdot; \bar{b}')$ is differentiable with $y_1(\bar{b}'; \bar{b}') > 1$ for all $\bar{b}' > b^*$.⁷

Thus far, we have established that the slope of the cut-off $y(\cdot)$ is zero approaching the risk-free limit b^* from the left, but greater than one approaching b^* from the right. Therefore, the cut-off has a good kink at b^* , and it has no differentiable upper support function at this point. This means we have established:

Lemma 3. *At every $\bar{b}' \neq b^*$, there exists a differentiable upper support function $\bar{y}(\cdot; \bar{b}')$ for $y(\cdot)$, and hence a differentiable lower support function $\underline{q}(\cdot; \bar{b}')$ for $q(\cdot)$. Moreover, $y(\cdot)$ has a good kink at b^* with $0 = y'(b^*-) < 1 < y'(b^*+)$.*

To construct a differentiable lower support function for W , we begin by constructing a differentiable lower support function for $V(b', x')$. However, this time, we use a lazy borrower's value function that differs from the pessimistic value function used to construct (13). The lazy borrower correctly anticipates x' , but incorrectly anticipates b' to be \bar{b}' . He takes on a debt of $\bar{b}''(x') = \hat{b}''(\bar{b}', x')$ independently of his previous obligation of b' . His value function is

$$M(b', x'; \bar{b}') = u(x' - b' + q(\bar{b}''(x'))\bar{b}''(x')) + \beta W(\bar{b}''(x')). \quad (15)$$

This means that,

$$\underline{W}(b'; \bar{b}') = A(x') + \int_{\bar{y}(b'; \bar{b}')}^{x^{\max}} [M(b', x'; \bar{b}') - A(x')] f(x') dx' \quad (16)$$

is a lower support function for W at \bar{b}' . We would like to establish that $\underline{W}(\cdot; \bar{b}')$ is differentiable. First, $M(\cdot, x'; \bar{b}')$ is continuously differentiable for all (x', \bar{b}') . Second, we note that without loss of generality, we may assume some optimal policy $\hat{b}''(\cdot, \cdot)$ is measurable, and hence the resulting lazy policy $\bar{b}''(\cdot)$ is also measurable.⁸ Third, the measurability of the lazy policy implies that $M_1(b', \cdot; \bar{b}')$ is measurable for all (b', \bar{b}') . Moreover, it is possible to show that $M_1(b', \cdot; \bar{b}')$ is uniformly bounded for all b' in some open neighbourhood of \bar{b}' . Hence the Leibniz rule for differentiating under the integral sign implies that $\underline{W}(\cdot; \bar{b}')$ is differentiable at $b' = \bar{b}'$ with⁹

$$\underline{W}_1(\bar{b}'; \bar{b}') = \int_{\bar{y}(b'; \bar{b}')}^{x^{\max}} M_1(\bar{b}', x'; \bar{b}') f(x') dx'. \quad (17)$$

⁷ Applying the implicit function theorem to (14) gives

$$\bar{y}_1(\bar{b}'; \bar{b}') = \frac{u_1(\bar{c}'(\bar{b}', y(\bar{b}')))}{u_1(\bar{c}'(\bar{b}', y(\bar{b}')) - u_1(y(\bar{b}') - s)} > 1.$$

⁸See the Measurable Maximum Theorem in [Aliprantis and Border \(2006, Theorem 18.19\)](#).

⁹ See for example [Weizsäcker \(2008, Theorem 4.6\)](#).

This means we have established:

Lemma 4. *At every \bar{b}' , there exists a differentiable lower support function $\underline{W}(\cdot; \bar{b}')$ for W .*

Combining the differentiable support functions for q and W from [Lemma 4](#) and [Lemma 3](#), we conclude that for all (b, x) and all $\bar{b}' \neq b^*$, the function

$$b' \mapsto u(x - b + \underline{q}(b'; \bar{b}')b') + \beta \underline{W}(b'; \bar{b}') \quad (18)$$

is a differentiable lower support function for $\phi(\cdot; b, x)$ at \bar{b}' .

We now have all the ingredients ready to apply the recipe.

Theorem 3. *Suppose $\hat{c}(\cdot, \cdot)$ and $\hat{b}'(\cdot, \cdot)$ are optimal policies for (11).¹⁰ Fix any state (b, x) and let $\hat{b}' = \hat{b}'(b, x)$. Then either*

1. $\hat{b}' = b^*$, or
2. $\hat{b}' < b^*$ and \hat{b}' satisfies the first-order condition

$$u_1(\hat{c}(b, x)) = \beta W_1(\hat{b}') = \int u_1(\hat{c}(b', x'))f(x') dx', \quad (19)$$

and the value function W is differentiable at \hat{b}' , or

3. $\hat{b}' > b^*$ and \hat{b}' satisfies the first-order condition

$$u_1(\hat{c}(b, x))(q(\hat{b}') + q_1(\hat{b}')\hat{b}') = \beta W_1(\hat{b}'), \quad (20)$$

and the nested functions W , q and y are differentiable at \hat{b}' with

$$W_1(\hat{b}') = \int_{y(\hat{b}')}^{x^{max}} u_1(\hat{c}(\hat{b}', x'))f(x') dx', \quad (21)$$

$$q_1(\hat{b}') = -\beta F_1(y(\hat{b}'))y_1(\hat{b}'), \quad (22)$$

$$y_1(\hat{b}') = \frac{u_1(\hat{c}(\hat{b}', y(\hat{b}')))}{u_1(\hat{c}(\hat{b}', y(\hat{b}'))) - u_1(y(\hat{b}') - s)}. \quad (23)$$

Proof. The first case does not involve any first-order conditions. We apply the recipe for the other two cases.

Step 1. At the optimal choice $\hat{b}' = \hat{b}'(b, x)$, the borrower's objective $\phi(\cdot; b, x)$ has a trivial upper support function $U(b'; b, x) = \phi(b', b, x)$.

Step 2. At \hat{b}' , the borrower's objective $\phi(\cdot; b, x)$ is sandwiched between (18) and $U(\cdot; b, x)$. So, the Differentiable Sandwich Lemma ([Lemma 1](#)) implies the borrower's objective is differentiable at the optimal debt choice \hat{b}' .

¹⁰ Note that c does not appear in (11). It is shorthand for $x - b + q(b')b'$.

Step 3. Repeated application of Reverse Calculus ([Lemma 2](#)) implies that W (in the second case) or W , q and y (in the third case) are differentiable at \hat{b}' .¹¹ \square

The first-order condition (20) can be interpreted as follows. The borrower equates the marginal benefit of owing debt with the marginal cost. The marginal cost consists of the expected marginal utility of the foregone consumption when repaying the following period (when the endowment shock is above the default cut-off). The marginal benefit consists of the marginal utility of consumption times the marginal revenue from promising an extra payment to the lender. Reverse calculus allows us to quantify the marginal revenue of promises. Specifically, when promising an extra payment, the honour probability decreases according to q_1 , which reflects an increase in the default cut-off of y_1 .

Also note that the theorem gives formulae for all derivatives. In particular all prices and marginal prices can be written in terms of quantities. This means it is possible to write the first-order condition in terms of quantities only, which can be helpful in computational work.¹²

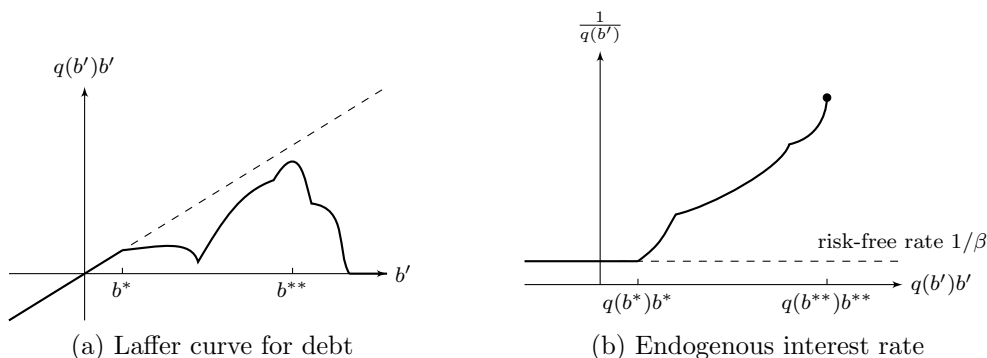


Figure 5: Characterization of endogenous borrowing

Credit Limits. We now turn our attention to the borrower's behaviour near the credit limit. The amount the lender is willing to pay, $q(b')b'$ in return for a promise of b' is not an increasing function. This is because there are two types of empty promises: $b' = 0$, and b' so large it is never honoured. The borrower's return on promises therefore follows a Laffer curve, depicted in [Figure 5a](#). The borrower's credit limit is the maximum of this curve,

¹¹ We repeatedly apply [Lemma 2](#) as follows. First, we apply rule (i) (summation) to (12a) to establish that both terms, $b' \mapsto u(x - b + q(b')b')$ and $b' \mapsto \beta W(b')$ are differentiable at \hat{b}' . Hence W is differentiable at \hat{b}' . If $\hat{b}' < b^*$, then we know that $q(b') = \beta$, so we stop here, giving the second case. Otherwise we are in the third case, and we continue. We apply rule (iv) (function composition) to $b' \mapsto u(x - b + q(b')b')$, which establishes that $b' \mapsto q(b')b'$ is differentiable at \hat{b}' . Next, we apply rule (ii) (multiplication) to establish that q is differentiable at \hat{b}' . This also means that both sides of (12c) are differentiable. So we apply rule (iv) (function composition) to the right side, $b' \mapsto \beta[1 - F(y(b'))]$ and conclude that y is differentiable at \hat{b}' .

¹² A FOC without prices is $u_1(\hat{c}(b, x))[1 - F(y(\hat{b}')) - F_1(y(\hat{b}'))y_1(\hat{b}')\hat{b}'] = W_1(\hat{b}')$.

$q(b^{**})b^{**}$, where

$$b^{**} = \arg \max_{b'} q(b')b'. \quad (24)$$

We apply the recipe for this new optimization problem. If $b^{**} > b^*$, then we have already constructed a differentiable lower support function for q at b^{**} , so the Differentiable Sandwich Lemma (Lemma 1) together with the Reverse Calculus Lemma (Lemma 2) imply that q is differentiable at b^{**} with

$$q(b^{**}) + q_1(b^{**})b^{**} = 0. \quad (25)$$

Substituting this into the Euler equation (20), we see that the marginal benefit of taking on debt at b^{**} is zero, while the marginal cost is positive. Therefore, we conclude

Theorem 4. *For any given model primitives, either*

1. *the overall and risk-free credit limits coincide, i.e. $b^{**} = b^*$, or*
2. *the overall credit limit is higher and exhausting it is suboptimal, i.e. $b^{**} > b^*$ and $\hat{b}'(b, x) < b^{**}$ for all states (b, x) .*

This conclusion is a logical generalization of behaviour in Aiyagari's (1994) model. Both here and there, the borrower reaches the risk-free credit limit with positive probability. In the model we study, the overall credit limit is potentially higher, as the borrower has the additional possibility of taking out risky loans. However, behaviour near the two credit limits is strikingly different. Below the risk-free limit, the interest rate $1/q(b')$ remains constant as the loan size $q(b')b'$ increases. Above the risk-free limit, the interest rate increases as the borrower takes on more debt and increases the default risk, as depicted in Figure 5b. This difference accounts for why borrowers might exhaust their risk-free limit, but not their overall limit.

Arellano (2008, Figure 2) plots a similar Laffer curve as in Figure 5a. Possibly for computational reasons, her curve is smooth and does not depict the upward kink of the Laffer curve at the risk-free limit, b^* . She does not apply first-order conditions along the Laffer curve.

Final Remarks. Our main contributions here are: (i) establishing that first-order conditions – involving derivatives of both policy functions and value functions – hold at optimal choices, (ii) providing formulas for these derivatives, so (marginal) prices can be written in terms of quantities, and (iii) applying the same logic to a Laffer curve for debt, concluding that borrowers do not exhaust their credit limits.

Others have applied these techniques to study various issues in both consumer and sovereign credit markets; see Jeske, Krueger and Mitman (2013) and Müller, Storesletten and Zilibotti (2019).

To keep the analysis simple, we assumed that shocks are IID. This condition was important for Arellano (2008) to establish that the default policy is a cut-off rule. More generally, persistent shocks cause interest rates to depend on the shock in addition to the size of the loan, which is crucial for understanding how credit markets operate when borrowers are distressed.

There are two potential ways to accommodate persistent shocks. First, [Chatterjee et al. \(2007, Theorem 3\)](#) established that two-sided cut-off rules are optimal in an environment with persistent shocks. We conjecture that it is possible to construct differentiable support functions for the two cut-offs, and use this to construct a differentiable upper support function for the repayment probability.

Second, we could assume that all persistence is summarized by a finite Markov chain. The shock x would be drawn from a distribution determined by a Markov state, such as “boom” or “recession.” There would be one cut-off rule for each Markov state, and hence one risk-free credit limit for each state. Hence there would be multiple debt-levels where first-order conditions are inapplicable.

3.3. Convex Dynamic Programming

[Benveniste and Scheinkman \(1979\)](#) study *value* functions in smooth convex dynamic programming problems, but not *policy* functions. Their main theorem establishes that value functions in this setting are differentiable. The Differentiable Sandwich Lemma leads to an elementary proof of their theorem.

Problem 1. Consider the following dynamic programming problem:

$$V(c) = \sup_{c' \in \{\bar{c}' : (c, \bar{c}') \in \Gamma\}} u(c, c') + \beta V(c'), \quad (26)$$

where the domain of V is C . We assume that (i) Γ is a convex subset of $C \times C$, (ii) u is concave, and (iii) $u(\cdot, c')$ and $u(c, \cdot)$ are differentiable, respectively.

Theorem 5 (Benveniste-Scheinkman Theorem). *If \hat{c}' is an optimal choice at state $c \in \text{int}(\{\bar{c} : (\bar{c}, \hat{c}') \in \Gamma\})$, then V is differentiable at c with $V_c(c) = u_c(c, \hat{c}')$.*

Proof. V is concave because u is concave and Γ is convex. Hence, the supporting hyperplane theorem can be applied to the hypograph of V to construct a linear upper support function U that touches V at c . We construct the differentiable lower support function $L(c) = u(c, \hat{c}') + \beta V(\hat{c}')$. [Lemma 1](#) delivers the conclusions. \square

Graduate economics textbooks such as [Stokey and Lucas \(1989\)](#) do not provide a self-contained proof of this theorem. Our proof is short and elementary, and therefore suitable for junior graduate students. The original proof is based on a sandwich lemma, which [Benveniste and Scheinkman \(1979\)](#) prove with the help of [Rockafellar \(1970, Theorem 25.1\)](#). (Their lemma imposes a redundant assumption, that the lower support function be concave.) [Mirman and Zilcha \(1975, Lemma 1\)](#) prove a one-dimensional special case using Dini derivatives rather than sandwiches.

3.4. Non-Convex Dynamic Programming

In this section, we study a more general class of stochastic dynamic programming problems. We drop all convexity assumptions, and we accommodate discrete choice sets and uncertainty. As discussed in the introduction, discrete choices can lead to kinks, and the

expectations operator can hide problematic kinks. Nevertheless, we establish that recursive first-order conditions hold at optimal choices. This is true for two reasons. First, discrete choices only ever introduce bad kinks – it is always feasible for the decision maker to stick with a particular discrete choice when the state changes, so any deviation from this must be favourable. Second, since all kinks point in the same direction (i.e. all bad), kink cancellation does not occur when calculating expectations.

Problem 2. Each period, a household chooses $(c', d') \in \Omega$, where $c' \in \mathbb{R}^n$ are *continuous* choices and d' are *discrete* choices. The state variable consists of the previous period's choices (c, d) as well as an exogenous Markov state $\theta \in \Theta$ with transition matrix π . We assume that the per-period utility $u(c, c'; d, d'; \theta)$ is differentiable with respect to c and c' . The set of combinations of states and feasible choices is denoted Γ . The Bellman equation is

$$V(c, d, \theta) = \sup_{c', d'} u(c, c'; d, d'; \theta) + \beta \sum_{\theta' \in \Theta} \pi_{\theta\theta'} V(c', d', \theta'),$$

s.t. $(c, c'; d, d'; \theta) \in \Gamma$,

where the domain of V is $\Omega \times \Theta$.

Suppose that $\omega(c, d, \theta)$ is an optimal choice vector at the state (c, d, θ) .

Definition 1. Fix any state (c, d, θ) , and consider the optimal choices $(\bar{c}', \bar{d}') = \omega(c, d, \theta)$ and $(\bar{c}''(\theta'), \bar{d}''(\theta')) = \omega(\bar{c}', \bar{d}', \theta')$ for the following two periods. Then the set of feasible one-shot deviations from \bar{c}' is

$$\Lambda(c, d, \theta) = \{c' : (c, c'; d, \bar{d}'; \theta) \in \Gamma, \text{ and for all } \theta', (c', \bar{c}''(\theta'); d', \bar{d}''(\theta'); \theta') \in \Gamma\}.$$

Theorem 6. Let $(\hat{c}', \hat{d}') = \omega(c, d, \theta)$ be optimal choices at state (c, d, θ) . If \hat{c}' is an interior choice, i.e. $\hat{c}' \in \text{int}(\Lambda(c, d, \theta))$, then (i) $V(\cdot, \hat{d}')$ is differentiable at \hat{c}' and (ii) \hat{c}' satisfies the first-order condition

$$-u_{c'}(c, \hat{c}'; d, \hat{d}'; \theta) = \beta \sum_{\theta'} \pi_{\theta\theta'} V_c(\hat{c}', \hat{d}', \theta') = \beta \sum_{\theta'} \pi_{\theta\theta'} u_c(\hat{c}', \hat{c}''(\theta'); \hat{d}', \hat{d}''(\theta'); \theta'),$$

where $(\hat{c}''(\theta'), \hat{d}''(\theta')) = \omega(\hat{c}', \hat{d}', \theta')$.

Proof. We assumed that \hat{c}' maximises

$$\phi(c') = u(c, c'; d, \hat{d}'; \theta) + \beta \sum_{\theta' \in \Theta} \pi_{\theta\theta'} V(c', \hat{d}', \theta'), \quad (27)$$

where the domain of ϕ is $\Lambda(c, d, \theta)$.

Ingredients. We prepare the ingredient support functions. V has a differentiable lower support function at $(\hat{c}', \hat{d}', \theta')$,

$$v(c', \hat{d}', \theta') = u(c', \hat{c}''(\theta'); d', \hat{d}''(\theta'), \theta') + \beta \sum_{\theta'' \in \Theta} \pi_{\theta'\theta''} V(\hat{c}''(\theta'), \hat{d}''(\theta'), \theta''). \quad (28)$$

This leads to a differentiable lower support function for ϕ at \hat{c}' , namely

$$L(c') = u(c, c'; d, \hat{d}'; \theta) + \beta \sum_{\theta' \in \Theta} \pi_{\theta\theta'} v(c', \hat{d}', \theta').$$

Since we assumed \hat{c}' is an interior choice, v and L are well-defined in an open neighbourhood of \hat{c}' .

Step 1. ϕ has a differentiable upper support function at \hat{c}' , namely $U(c') = \phi(\hat{c}')$.

Step 2. ϕ is sandwiched between L and U at \hat{c}' . By the Differentiable Sandwich Lemma, ϕ is differentiable at \hat{c}' with $\phi_{c'}(\hat{c}') = U_{c'}(\hat{c}') = 0$.

Step 3. The addition rule of Reverse Calculus implies that V is differentiable with respect to c' at each $(\hat{c}', \hat{d}', \theta')$. \square

Milgrom and Segal (2002, Corollary 2) proved a special case of this theorem in a static context (when $\beta = 0$), and with two redundant assumptions, namely that the utility function is equidifferentiable and the marginal utilities are uniformly bounded. Milgrom and Segal make good use of these assumptions in establishing global differentiability properties of value functions in their Theorem 3. But these assumptions are redundant for studying first-order conditions.

Both conditions are problematic. The equidifferentiability condition is difficult to check in infinite horizon dynamic problems. More importantly, the uniformly bounded derivative condition conflicts with Inada conditions. Inada conditions are often imposed to ensure first-order conditions hold by directing optimal choices away from boundaries.

Milgrom and Segal (2002) impose these conditions in order to ensure that the value function's directional derivatives exist globally. Similarly, other papers make assumptions including Lipschitz continuity (Clarke (1975)) and supermodularity (Amir *et al.* (1991)) to ensure the existence of directional derivatives. Our recipe does not make use of directional derivatives, so we can dispense with these assumptions.

4. Conclusion

We can not assume that the solutions to nested optimization problems are smooth. Nevertheless, we found that in some economic problems, it is optimal for decision makers to choose from smooth parts of their menus. Nested optimization problems come in many forms, so we do not provide a one-size-fits-all theorem. Instead, we devised a recipe based on the Differentiable Sandwich Lemma. Applying the recipe requires finding appropriate differentiable support functions.

There are potentially many ways to mix and match different constructions of upper and lower halves of sandwiches. We used five constructions throughout, namely (i) horizontal lines above maxima, (ii) supporting hyperplanes above concave functions, (iii) Reverse Calculus, (iv) lazy value functions below rational value functions, and (v) pessimistic cut-off rules. Of these constructions, only the Reverse Calculus construction is truly unprecedented. The power of our approach derives from the ability to combine these constructions, and the

three-step recipe provides an intuitive way to organize them. For example, the unsecured credit application uses all but the supporting hyperplane construction.

There are other possibilities that we did not explore. Decision makers can be “lazy” in ways that lead to upper support functions, such as being lazily optimistic about future opportunities. In bargaining games, a lower support function for one player’s value function leads to an upper support function for the other player’s value function.

Appendix A. Further Reverse Calculus Rules

This appendix provides two further Reverse Calculus rules that were not used in the paper, but might be useful for other problems. Specifically, the rules relate to convex combinations and function composition.

The rule for convex combinations is complicated, because the forward calculus step is not obvious. The following lemma incorporates both a forward and reverse calculus result.

Lemma 5. *Suppose $E : C \rightarrow [0, 1]$, $F : C \rightarrow \mathbb{R}$, and $G : C \rightarrow \mathbb{R}$ have differentiable lower support functions e , f , and g respectively at \bar{c} . Consider the function*

$$H(c) = E(c)F(c) + (1 - E(c))G(c).$$

If $F(\bar{c}) > G(\bar{c})$, then

1. *The function $h(c) = e(c)f(c) + (1 - e(c))g(c)$ is a differentiable (local) lower support function for H at \bar{c} .*
2. *If H is differentiable at \bar{c} , then e , f , and g are also differentiable at \bar{c} .*

Proof. Consider the two functions,

$$\begin{aligned} h(c) &= e(c)f(c) + (1 - e(c))g(c) \\ \tilde{h}(c) &= E(c)f(c) + (1 - E(c))g(c). \end{aligned}$$

Since $f(\bar{c}) > g(\bar{c})$, we have that $h(c) \leq \tilde{h}(c)$, and hence $h(c) \leq H(c)$ in some open neighbourhood of \bar{c} . This establishes part (i).

For part (ii), we see that \tilde{h} is differentiable sandwiched between h and H at \bar{c} . By the Differentiable Sandwich Lemma, \tilde{h} is differentiable at \bar{c} . This implies $E(c) = [\tilde{h}(c) - g(c)]/[f(c) - g(c)]$ is also differentiable at \bar{c} . Therefore, both terms of H , namely $E(c)F(c)$ and $(1 - E(c))G(c)$, have differentiable lower support functions, $E(c)f(c)$ and $(1 - E(c))g(c)$, respectively. Part (i) of Lemma 2 implies that both terms are differentiable at \bar{c} , and hence F and G are differentiable at \bar{c} . \square

Finally, we consider function composition of two functions, neither of which are known to be differentiable a priori.

Lemma 6. *If $H(c) = J(K(c))$ is differentiable at \bar{c} , where*

- $J : \mathbb{R} \rightarrow \mathbb{R}$ has an inverse J^{-1} and a differentiable lower support function $j(\cdot)$ at $K(\bar{c})$,

- $K : \mathbb{R} \rightarrow \mathbb{R}$ has an inverse K^{-1} and a differentiable lower support function $k(\cdot)$ at \bar{c} , and
- $j'(K(\bar{c})) \neq 0$ and $k'(\bar{c}) \neq 0$,

then J and K are differentiable at $K(\bar{c})$ and \bar{c} respectively.

Proof. We assume without loss of generality that $j'(K(\bar{c})) > 0$.¹³ We now establish that this implies j^{-1} is a differentiable upper support function for J^{-1} . To see this, we evaluate the inequality $j(c) \leq J(c)$ at $J^{-1}(x)$ which gives

$$j(J^{-1}(x)) \leq J(J^{-1}(x)) = x.$$

Applying j^{-1} to both sides gives $J^{-1}(x) \leq j^{-1}(x)$.

We can express $K(\cdot)$ as a function of J and H as follows:

$$J^{-1}(H(c)) = J^{-1}(J(K(c))) = K(c).$$

This has a differentiable upper support function $j^{-1}(H(c))$ at \bar{c} . Thus K has differentiable upper and lower support functions at \bar{c} , and is therefore differentiable by [Lemma 1](#). Next, evaluating $H(c) = J(K(c))$ at $c = K^{-1}(x)$ gives

$$H(K^{-1}(x)) = J(K(K^{-1}(x))) = J(x),$$

so J is differentiable at $K(\bar{c})$ by the chain rule and inverse function theorem. \square

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¹³ If $j'(K(\bar{c})) < 0$, then the lemma can be applied to $\tilde{H}(c) = \tilde{J}(K(c))$, where $\tilde{H}(c) = -\frac{1}{H(c)+x_0}$ and $\tilde{J}(c) = -\frac{1}{J(c)+x_0}$, where x_0 is a suitable constant to prevent division by zero near $K(\bar{c})$. In this case, $\tilde{j}(c) = -\frac{1}{j(c)+x_0}$ is a lower support function for \tilde{J} with a strictly positive derivative $\tilde{j}'(c) = \frac{1}{[j(c)+x_0]^2} j'(c)$.

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